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# SEMIGROUP APPROACH FOR THE SOLUTION OF BOUNDARY LAYER EQUATION WITH SINC FUNCTION TERM

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## Abstract

*Boundary layer equation is crucial in fluid dynamics for modeling viscous flow near surfaces. It provides insight into flow behaviour, drag reduction, heat transfer and stability, making it essential in both theoretical and applied fluid mechanics. This paper examines the semigroup approach for solving the boundary layer equation incorporating a Sinc function term.The addition of the Sinc function term necessitates specialized functional analysis techniques and its effects are analyzed. We establish well-posedness in a suitable function space by analyzing the existence, uniqueness of the mild solution. We demonstrate the semigroup method's effectiveness in capturing boundary layer dynamics, contributing to the study of semigroup methods in fluid mechanics and partial differential equations. The influences of the Sinc function term are analyzed and illustrative examples are included to validate the approach and to also highlight its applicability.*

**Key words:** Boundary layer equation, Banach Spaces, Sinc function, Strongly Continuous Semigroup, Mild solution.

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## 1. INTRODUCTION

The boundary layer equation plays a crucial role in fluid dynamics, describing the behavior of viscous fluid flow near solid surfaces. Prandtl [1] initiated the boundary layer theory, significantly advancing the study of viscous flows. This equation has been extensively studied due to its in aerodynamics, heat transfer, and hydrodynamics stability. The traditional method for solving the boundary layer equation include similarity transformations, perturbation methods, and numerical simulations. However, in recent years, the semigroup approach has emerged as a powerful tool for the existence, uniqueness, and stability of solutions to partial differential equations (PDEs). This method leverages the properties of operator semigroups to study the time evolution of solutions in appropriate functional spaces. Semigroup theory provides a robust framework for studying time-dependent PDEs. Hille and Phillips [2] laid the foundation of semigroup analysis, later extended by Pazy [3] to cover evolutionary equations in Banach spaces. The semigroup approach has been successfully applied to fluid dynamics problems, including the Navier-Stokes equations [4]and parabolic PDEs [5]. Henry [6] applied semigroup method for nonlinear equations where he showed that the mild solution method could be used to establish the well-posedness of these equations in various function spaces.

Recently, the work of Prüss [7] extended the semigroup approach to treat boundary boundary value problems with non-homogeneous conditions, which arise in practical fluid dynamics applications. Husssian and Kato [8] explored the use of semigroup approach to solve boundary layer equation numerically. Their research showed that semigroup-based numerical methods offer significant advantages in terms of stability compared to other methods like finite difference schemes, particularly for complex flow configurations and high Renolds numbers.

The Sinc function and its interpolation techniques have been widely studied in numerical analysis [10, 11,12]. The function is well known for its applications in signal processing and numerical analysis, introducing oscillatory behavior into the equation, which affects the solution properties. Also, the function's unique properties such as band-limited representation and rapid decay, makes it useful for solving differential equations. Lund and Bowers [10] have explored Sinc-base methods for approximating PDE solutions, but their interaction with semigroup methods remains an open area of research. This study aims to bridge this gap by developing a semigroup-theoretic approach for analyzing the well-posedness of boundary layer equation with Sinc function term. The integration of Sinc function term in the equation introduces additional complexities, requiring additional mathematical techniques for analysis and solution. Our approach focuses on establishing well-posedess of the equation with the inclusion of the Sinc function term.

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## 2. PRELIMINARIES

In this section, we introduce the fundamental mathematical concepts and notations required for our analysis of the boundary layer equation with a Sinc function term using the semigroup approach.

### 2.1. Functional Space and Operators

Let $X$ be a Banach space, and consider a differential operator $A$ on a dense subspace $D(A) \subset X$. We work within the frame work of semigroup theory, requiring the following definitions:

2.2. **Normed and Banach Spaces:** A normed space $(X,\|\cdot\|)$ is a vector space if it is complete with respect to this norm. A normed space $(X,\|\cdot\|)$ is called **Banach space** if every Cauchy sequence in $(X,\|\cdot\|)$ converges.

2.3. **Hilbert Space:** A special case of Banach space where the norm is induced by an inner product $\langle \cdot,\cdot \rangle$.

We primarily consider function spaces such as $L^p(\Omega)$ and Sobolevspaces $H^k(\Omega)$, which are crucial in studying PDEs [ 5, 13, 14, 21].

## 3. BOUNDARY LAYER EQUATION WITH SINC TERM

The general form of the boundary layer equation with a Sinc function term can be written as:

$$\frac{\partial u}{\partial t} + Au = f(x,t) + S(x,t) \qquad (1)$$

were :

- $u(x,t)$ represents the velocity field,
- $A$ is a differential operator capturing the viscous effects,
- $f(x,t)$ is an external forcing term,

- $S(x,t)$ is a Sinc function-modulated term affecting the solution structure.

The boundary conditions and initial conditions depend on the physical context.

## 4. SEMIGROUP THEORY

Let $X$ be a Banach space, and let $A: D(A) \subset X \to X$ be a densely defined, closed linear operator. A strongly continuous semigroup (or $C_0$ −Semigroup, or a semigroup of class $C_0$) $T(t)_{t\geq 0}$ on the Banach space $X$ is a family of bounded linear operators such that

$$\begin{cases} T(0) = I \quad (\text{the identity operator}) \\ T(s+t) = T(s)T(t) \text{ for all } t,s \geq 0 \\ \lim_{t\to 0} T(t)x = x \quad \text{ for all } x \in X \end{cases} \tag{2}$$

The generator of the semigroup, denoted by $A$, is defined as

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t},$$

whenever the limit exists. The well-posedness of the equation depends on whether $A$ generates a semigroup on $X$ [3, 18, 19, 22, 24].

### 4.1. Spectral Properties of the Operators:

The spectrum of $A$ denoted by $\sigma(A)$, plays a crucial role in determining the stability of the solutions. The resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ helps analyze whether $A$ an exponentially stable semigroup.

### 4.2. Sinc Function and its Properties

The Sinc function is define as

$$\text{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \qquad x \neq 0, \ \text{Sinc}(0) = 1 \tag{3}$$

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It has useful properties such as rapid decay and interpolation capabilities, making it a valuable tool in numerical method and spectral analysis [12].

## 4.3. Mild Solution through Semigroup Approach:

A mild solution to a differential equation is a solution that is defined through an integral equation involving a semigroup. For an evolution equation of the form

$$\frac{du}{dt} + Au = f(t),$$

the mild solution is obtain using Duhamel's formula and is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s) ds \qquad (4)$$

where $T(t)$ is the semigroup generated by the operator $A$ and $u_0$ is the initial condition.The term mild is used because the solution is often less regular than classical solutions but still provides valuable information.

Also, if $T(t)$ is strongly continuous semigroup generated by $A$, then the mild solution can be expressed as

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\big(f(s) + S(s)\big) ds \qquad (5)$$

Equation $(5)$ provides insight into the existence, uniqueness, and stability of the solution, See [23, 24].

## 4.4. Mathematical Formulation of the Problem

We begin by representing the boundary layer equation as follows

$$\frac{\partial u}{\partial t} + Au = f(x,t) + S(x,t), \quad x \in \Omega,\ t \geq 0 \qquad (6)$$

where $u(x,t)$ represent the unknown function (the state of the system) at time $t$ and position $x$ , $A$ is a differential operator (often related to the Laplacian or similar operator that models

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diffusion or advection in fluid dynamics), $f(x,t)$ and $S(x,t)$ are external forcing terms. The initial condition for this equation is:

$u(x,0)=u_0(x), \quad x\in\Omega$, where $u_0(x)$ represents the state of the system at time $t=0$.

## 5. Semigroup Representation and Solution Method

The solution is sought in the layout of semigroup theory, which provides a powerful method to deal with evolution equations. We assume that $A$ generates a strongly continuous semigroup $T(t)_{t\geq 0}$ on a Banachspace $X$. The semigroup approach allows us to express the solution in the form of equation $(5)$. Here, the term $T(t)u_0$ represents the evolution of initial condition, and the integral term capture the influence of the forcing $f(x,t)+S(x,t)$ over time, See [18, 24].

### 5.1. Well-Posedness of the Problem

To ensure the problem is well-posed, we need to establish that equation $(5)$ exists, is unique, and depends continuously on the initial conditions. This is done through the following steps:

- **Existence:** By the properties of semigroups, the integral equation for $u(t)$ is well-defined and yields a solution. The regularity of $f(x,t)$ and $S(x,t)$ ensure that the integral is finite and the solution ie well-behaved.
- **Uniqueness:** If two solutions $u_1(t)$ and $u_2(t)$ exist, we show that $u_1(t)=u_2(t)$ by employing the Banach fixed-point theorem [3, 25], this relying on the fact that the operator $T(t)$ is strongly continuous and the forcing terms $f(x,t)$ and $S(x,t)$ are assume be identical for both solutions.
- **Continuous Dependence:** Since the solution depends continuously on the initial condition $u_0(x)$, small changes in the initial condition lead to small changes in the solution, which is a key property in proving stability.

### 5.2. Influence of the SincFunctionTerm

A distinctive feature of this research is the inclusion of Sinc function in the forcing term. The Sinc function introduce oscillations into the system, which can affect both the regularity and stability of the solution. In the context of boundary layers, the Sinc function term might represent oscillatory external forces or disturbances that interact with the boundary layer dynamics.

- **Oscillatory Behavior:** The Sinc function's oscillations may induce transient effects in the system, but under suitable condition (e.g., decaying forcing terms), these oscillations do not lead to unbounded growth of the solution.
- **Damping Effects:** The decay of the Sinc function ensures that its influence fades over time, which implies that the system will eventually settle to a steady state, as shown by the exponential decay established earlier.

### 5.3. Regularity of Solutions

This is another crucial aspect of the research. Regularity refers to the smoothness of the solution. If $u(t)$ belongs to a function space with sufficient differentiability (e.g., $C^1, L^2$, etc.), it is said to be regular. The higher the regularity the smoother the solution, and this property often plays a crucial role in the stability and long-term behavior of solutions. The solution $u(t)$ is shown to belong to the class of $C^1$ under suitable regularity assumptions on the initial data $u_0$ and the forcing term $f(x,t)+S(x,t)$. Higher regularity ensures that the solution behaves smoothly over times and that its derivatives exist and are continuous.

By applying standard semigroup theory, we can also show that higher derivatives of $u(t)$ exist and are bounded, which is important for numerical simulations and further theoretical analysis, See [18, 19, 22].

## 6. RESULTS

**Well-Posedness of Boundary Layer Equation:** We consider the initial-boundary value problem:

$$\frac{\partial u}{\partial t} + Au = f(x,t) + S(x,t),\ x \in \Omega,\ t > 0$$
$$u(x,0) = u_0(x),\ x \in \Omega \tag{7}$$

where $A$ is as assumed in section 4.

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**Theorem 5.1** (Existence and Uniqueness of Mild Solution): If $A$ is as assumed in section 5, and if $f(x,t)$ and $S(x,t)$ satisfy suitable regularity conditions, then equation $(7)$ has a unique mild solution given by equation $(5)$.

Proof:

We apply Duhamel's formula [3, 15] to construct a mild solution of equation $(5)$. Since the assumption of section 5 holds, it satisfies;

- $T(0)=I,$
- $T(t+s)=T(t)T(s)$
- $|T(t)|\leq Me^{\omega t}$ for some $M,\omega\geq 0$

The existence of the integral form follows from the properties of semigroups and the assumed regularity of $f(x,t)$ and $S(x,t)$, ensuring that the function inside the integral is well-defined.

Next, we prove the uniqueness of the solution. Assume there exist two solutions $u_1(t)$ and $u_2(t)$ satisfying the integral equation

$$u_1(t)-u_2(t)=T(t)\big(u_1(0)-u_2(0)\big)+\int_0^t T(t-s)\big(f_1(s)-f_2(s)+S_1(s)-S_2(s)\big)ds$$

If $u_1(0)=u_2(0)$ and $f_1=f_2, S_1=S_2$, then

$$\|u_1(t)-u_2(t)\|\leq M\int_0^t e^{\omega(t-s)}\|0\|ds=0$$

Thus, $u_1=u_2$, proving the uniqueness.

Finally, we prove the continuity of the solution. By the strong continuity of $T(t)$, we can show that $u(t)$ is continuous in $X$ as follows:

By assumption, $T(t)$ is strongly continuous semigroup, meaning that for all $u_0\in X,$

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$$\lim_{x\to 0^+} T(t)u_0 = u_0 \in X$$

This directly implies that $T(t)u_0$ is continuous as function of $t \in X$ .

Now, we need to prove the continuity of integral term by analyzing the integral term:

$$I(t)=\int_0^t T(t-s)\big(f(s)+S(s)\big)ds$$

To prove that $I(t)$ is continuous in $X$, we consider a small perturbation $h$ in time and examine the difference:

$$I(t+h)-I(t)=\int_0^{t+h} T(t+h-s)\big(f(s)+S(s)\big)ds-\int_0^t T(t-s)\big(f(s)+S(s)\big)ds$$

Splitting the difference, we get

$$I(t+h)-I(t)=\int_0^t \big(T(t+h-s)-T(t-s)\big)\big(f(s)+S(s)\big)ds+$$

$$\int_t^{t+h} T(t+h-s)\big(f(s)+S(s)\big)ds$$

For the first term:

$$\int_0^t \big(T(t+h-s)-T(t-s)\big)\big(f(s)+S(s)\big)ds$$

Since $T(t)$ is strongly continuous, for each fixed $s$,

$$\lim_{h\to 0} T(t+h-s)=T(t-s) \text{ in } X.$$

Since $f(s)+S(s)$ is integrable, we can use the Lebesgue Dominated Convergence Theorem [16, 17] to conclude that:

$$\lim_{h\to 0}\int_0^t \big(T(t+h-s)-T(t-s)\big)\big(f(s)+S(s)\big)ds=0.$$

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For the second term.

$$\lim_{h \to 0} \int_t^{t+h} T(t+h-s)\left(f(s)+S(s)\right)ds$$

As $h \to 0$, the interval $[t, t+h]$ shrinks to zero, and since $f(s)+S(s)$ is integrable, this integral also vanishes.

Finally, since both terms vanish as $h \to 0$, we have

$$\lim_{h \to 0} \left\| I(t+h) - I(t) \right\| = 0$$

Thus, $I(t)$ is continuous in $X$, and their sum $u(t)$ is also continuous in $X$.

Therefore, the mild solution $u(t)$ in equation $(5)$ is continuous in $X$. □

**Proposition 5.2.** (Decay Rate of Solution): If $f(x,t)$ and $S(x,t)$ vanishes as $t \to \infty$, then $|u(t)| \to 0$ exponentially fast.

Proof:

Since $f(x,t)$ and $S(x,t)$ tend to zero, the integral term in

$$\|u(t)\| \le Me^{-\lambda t}\|u_0\| + M\int_0^t e^{-\lambda(t-s)}\|f(s)+S(s)\|ds$$

vanishes as $t \to \infty$. Thus,

$$\|u(t)\| \le Me^{-\lambda t}\|u_0\| \to 0$$

This shows exponential decay of solution. □

**Theorem 6.4.** (Higher Regularity of Solutions): If $u_0 \in D(A)$, and $f(x,t)+S(x,t)$ are sufficiently smooth, then the solution satisfies

$$\frac{du}{dt} + Au = f + S, \quad u \in C^1\left(\left(0,T\right), X\right).$$

Proof:

First, since $u_0 \in D(A)$, we apply $A$ to the mild solution formula

$$\frac{du}{dt} = AT(t)u_0 + A\int_0^t T(t-s)\left(f(s)+S(s)\right)ds + \left(f+S\right)$$

Using the properties of semigroups [3, 19],

$AT(t) = T(t)A$, we write

$$\frac{du}{dt} = Au + f + S$$

This implies that $u$ is differentiable with the stated regularity[6]. □

**Example 6.1.** Consider the boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{V}\dfrac{\partial^2 u}{\partial x^2} + \mathrm{sinc}(t), & x \in (0,1),\ t > 0 \\ u(t,0) = u(t,1) = 0, & t \geq 0 \\ u(0,x) = u_0(x) = \sin(\pi x). & \end{cases}$$

This is a linear boundary layer equation with Sinc forcing. We interpret the sinc term as an external **time-dependent forcing** that is spartially uniform (i.e., acts identically across all spatial points).

Let $A = \mathrm{V}\dfrac{d^2}{dx^2}$ with domain $D(A) = u \in H^2(0,1) \cap H_0^1(0,1)$. Then $A$ generates a **strongly continuous analytic semigroup** $T(t) = e^{tA}$.

The **mild solution** is given by:

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$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds.$$

The eigenfunctions of $A$ are:

$$e_n(x) = \sqrt{2}\sin(n\pi x), \qquad \lambda_n = -\mathrm{V}(n\pi)^2.$$

So the semigroup acts as:

$$T(t)u_0 = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle u_0, e_n\rangle e_n(x).$$

Since $u_0(x) = \sin(\pi x) = \frac{\sqrt{2}}{2}e_1(x)$, only the first mode is nonzero:

$$T(t) = e^{-\mathrm{V}\pi^2} \cdot \frac{\sqrt{2}}{2}e_1(x).$$

Now consider:

$$f(s,x) = \operatorname{sinc}(s)\cdot 1 = \operatorname{sinc}(s)\sum_{n=1}^{\infty}\langle 1, e_n\rangle e_n(x),$$

$$\langle 1, e_n\rangle = \sqrt{2}\int_0^1 \sin(n\pi x)\,dx = \begin{cases} \frac{2\sqrt{2}}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Hence:

$$T(t-s)f(s) = \sum_{\substack{n\ge 1 \\ n \text{ odd}}} e^{\lambda_n(t-s)\cdot\operatorname{sinc}(s)\cdot}\frac{2\sqrt{2}}{n\pi}\cdot e_n(e_n).$$

Integrating:

$$\int_0^t T(t-s)f(s)\,ds = \sum_{\substack{n\ge 1 \\ n \text{ odd}}}\left[\int_0^t e^{\lambda_n(t-s)}\cdot\operatorname{sinc}(s)\,ds\right]\cdot\frac{2\sqrt{2}}{n\pi}e_n(x).$$

Combining, we have the **mild solution**:

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$$u(t,x)=\frac{\sqrt{2}}{2}e^{-\mathrm{V}\pi^2 t}e_1(x)+=\sum_{\substack{n\geq 1\\ n\text{ odd}}}\left[\int_0^t e^{-\mathrm{V}(n\pi)^2(t-s)}\cdot\operatorname{sinc}(s)\,ds\right]\cdot\frac{2\sqrt{2}}{n\pi}e_n(x).$$

Truncating to first odd mode $(n=1)$, we have:

$$u(t,x)\approx\left[\frac{\sqrt{2}}{2}e^{-\mathrm{V}\pi^2 t}+\frac{2\sqrt{2}}{\pi}\int_0^t e^{-\mathrm{V}\pi^2(t-s)}\operatorname{sinc}(s)\,ds\right]\sin(\pi x).$$

This is a practical, computable **low-mode mild solution.**

**Example 6.2.** Consideranother case given by:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\mathrm{V}\dfrac{\partial^2 u}{\partial x^2}-u\dfrac{\partial u}{\partial x}+\phi(x)\operatorname{sinc}(t),\\ u(t,0)=u(t,1)=0,\ u(0,x)=\sin(\pi x),\\ \phi(x)=x(1-x).\end{cases}$$

This is a semilinear case with Sinc forcing. In this case, the **mild solution** is obtained as follows:

Let $F(u)(s)=-u(s)\partial_x u(s)+\phi\operatorname{sinc}(s)$, then:

$$u(t)=T(t)u_0+\int_0^t T(t-s)F(u(s))\,ds.$$

As in example 5.1,

$$T(t)u_0\approx\frac{\sqrt{2}}{2}e^{-\mathrm{V}\pi^2 t}\sin(\pi x).$$

For the nonlinear term

Assume $u(t,x)\approx\alpha(t)\sin(\pi x)$. Then:

- $\partial_x u=\alpha(t)\pi\cos(\pi x)$
- $u\partial_x u=\alpha^2(t)\pi\sin(\pi x)\cos(\pi x).$

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We project $u\partial_x u$ onto

$$e_1(x)=\sqrt{2}\sin(\pi x);$$

$$\langle u\partial_x u, e_1\rangle=\alpha^2(t)\pi\sqrt{2}\int_0^1 \sin^2(\pi x)\cos(\pi x)dx$$

$$\int_0^1 \sin^2(\pi x)\cos(\pi x)dx=\frac{1}{\pi}-\frac{1}{2\pi}=\frac{1}{2\pi}$$

So:

$$\langle u\partial_x u, e_1\rangle=\alpha^2(t)\pi\sqrt{2}\cdot\frac{1}{2\pi}=\frac{\sqrt{2}}{2}\alpha^2(t).$$

Projection of forcing term:

$$\langle \phi, e_1\rangle=\sqrt{2}\int_0^1 x(1-x)\sin(\pi x)dx=\sqrt{2}\cdot\frac{4}{\pi^3}.$$

Putting everything together, the scalar integral equation for $\alpha(t)$ is:

$$\alpha(t)=\frac{\sqrt{2}}{2}e^{-\nu\pi^2 t}+\int_0^t e^{-\nu\pi^2(t-s)}\left[-\frac{\sqrt{2}}{2}\alpha^2(s)+\frac{4\sqrt{2}}{\pi^3}\operatorname{sinc}(s)\right]ds.$$

This a **nonlinearVolterra equation of the second kind**.

## Remark:

- The nonlinear term creates a coupling effect where past values of $\alpha(s)$ impact present $\alpha(t)$.
- The integral equation can be approximated numerically (e.g., trapezoidal rule or fixed point iteration).
- The result is a reduced-order approximation to the original PDE using the semigroup eigenfunction method.

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## 7. CONCLUSION

We have successfully establish a pattern for solving boundary layer equations incoperatingSinc function in the forcing term using semigroup theory. The major results include the well-posedness, regularity and the decay rate of the solutions, along with detailed analysis of the role of the Sinc function term in these equations. This layout paves way for further exploration into numerical methods, such as the use of Sinc functions in discretization schemes, and also have applications in modeling physical systems with oscillatory boundary conditions. Illustrative examples have been shown to validate the approach and applicability.

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