ON STABILITY IN SEPARATIVE SEMIGROUP

Otobong J. Tom*
Fed. Uni. Tech., Ikot Abasi & Akwa Ibom State Uni., Ikot Akpaden

Otobong G. Udoaka
Department of Mathematics, Akwa Ibom State University, Ikot Akpaden

Ekere S. Udofia
Department of Mathematics, Akwa Ibom State University, Ikot Akpaden

Abstract

Stability, as introduced by Koch and Wallace (1956), has long stood as a central notion in semigroup theory, ensuring that inclusions of principal ideals collapse into equalities and that left and right structures align harmoniously. Separativity, on the other hand, generalises cancellativity while retaining algebraic regularity, and Burmistrovich's decomposition theorem revealed that every separative semigroup can be expressed as a semilattice of cancellative semigroups. Yet, whether separative semigroups inherit stability in the sense of Koch and Wallace has remained unresolved. In this work, we close this gap: we prove that semilattices of cancellative semigroups are stable, and hence every separative semigroup is inherently stable. This result elevates stability from a supplementary condition to a built-in feature of separative semigroups, offering a unified perspective that strengthens the foundations of semigroup theory and deepens its structural coherence.

Keywords: Stability in semigroup; Separative Semigroup; Cancellative semigroup; Semillatice decomposition; and Green's Relations.

1 Introduction

The concept of stability in semigroups was formally introduced by Koch and Wallace [16] in their seminal paper. A semigroup is said to be stable if for all $a, b \in S$:

$$aS \subseteq abS \implies aS = abS, \qquad Sa \subseteq Sab \implies Sa = Sab.$$

This algebraic definition reflects a form of "ideal stability" and has since become a cornerstone in the theory of semigroups, inspiring further structural investigations [1, 12]. Stability is important because it guarantees that the behaviour of principal ideals remains robust under multiplication. The study of stability has evolved, particularly through its connection with Green's relations, which classify semigroup elements based on their divisibility properties [16]. A crucial result from Koch and Wallace states that in a stable semigroup, Green's $\mathcal{D}-$ and $\mathcal{J}-$ equivalences coincide, implying that left and right ideal structures behave symmetrically. Anderson, et al [11] further expanded on this idea by studying stability conditions in various semigroup classes, highlighting its role in determining structural simplicity and regularity. The structure of quasi-separative semigroup was introduced by Drazin [12], where the connections between it and other semigroup properties, such as inversity, regularity, etc, were established. This was later extended by Krasilnikova and Novikove [22]. Parallel to this development, the notion of separativity was introduced to generalise cancellativity while preserving a degree of regularity. East and Higgings [9] explored Green's relation in greater depth, providing a more refined classification of semigroup elements' stability constraints. Hewitt and Zuckerman [6] applied separativity properties in their study, titled "l_1- algebra of a commutative semigroup", where they introduce harmonic analy sis on discrete commutative semigroups. Shourijeh [2] established the commutativity of separative semigroup and discussed its left presentation. Burmistrovich [7] provided a powerful structural result: a semigroup is separative if and only if it is isomorphic to a semilattice of cancellative semigroups. This theorem places separative semigroups in a broader algebraic context and has been extensively used in decomposition theory [13]. However, Burmistrovich's theorem itself does not discuss stability, leaving a gap in the understanding of whether separative semigroups inherit stability (in the sense of [16]). This gap motivates the present study. We aim to establish that separative semigroups are indeed stable in the sense of [19]. Our approach hinges on re-examining Burmistrovich's decomposition theorem from the perspective of stability. We first establish that a semilattice of cancellative semigroups is stable—a fact that has not been explicitly recorded in the literature. Since separative semigroups are precisely those semigroups isomorphic to such semilattices, it follows directly that every separative semigroup is stable.

2 Preliminaries

Definition 2.1 (Groupoid). *Let S be a non-empty set. Let $*$ be an operation such that $* : S \times S \to S$ be defined on S. Then $(S, *)$ is called* groupoid *if for all $a, b \in S, a * b \in S$.*

Definition 2.2 (Semigroup). *A groupoid is called* semigroup *if the binary operation is associative (i.e., for all $a, b, c \in S$, we have $(a * b) * c = a * (b * c)$).*

Definition 2.3 (Idempotent). *An element $e \in S$ is called an* idempotent *element if $e^2 = e$.*

Definition 2.4 (Cancellative semigroup). *A semigroup is called* cancellative *for all* $a, b, c \in S, ab = ac \implies a = b$(left-cancellative) *and* $ba = ca$ *(right-cancellative).*

For more about this, the reader is referred to [1, 4, 5, 7].

3 Separative semigroup

Definition 3.1. *A semigroup S is called* separative *if and only if the following two conditions hold for all $a, b \in S$:*

- $a^2 = ab$ *and* $ba = b^2$ *together imply* $a = b$, *and*
- $a^2 = ba$ *and* $ab = b^2$ *together imply* $a = b$.

The reader can also see [15]

Definition 3.2 (Qusi-separative). *A semigroup is called* quasi–separative *if* $a^2 = ab = ba = b^2 \implies a = b$, *[22].*

On a bright-line, one can say that a separative semigroup is a quasi-separative. Extracting from [20] we define a weakly separative semigroup when we have $asa = asb = bsa = bsb$ only if $a = b$ for all $a, b, s \in S$.

Theorem 3.1 (Proposition 1 of [12]). *If S is any quasi-separative semigroup, then, for all $a, b \in S$ we have*

$$a^2 = ab = b^2 \quad \text{if and only if} \quad a = b.$$

And the converse also holds.

Proof. If $a^2 = ab = b^2$, then we have

$$(ab)^2 = a^2b^2 = (aa)(bb) = a^2b^2,$$

and

$$(ab)(ba) = (aa)a(a) = a^4, \quad (ba)(ab) = (bb)b(b) = b^4.$$

Also,

$$(ba)^2 = b(ab)a = (bb)(aa) = b^2a^2,$$

and

$$a(ab) = a(aa)a = a^3, \quad b(ba) = b(bb)b = b^3.$$

Therefore,

$$(ab)(ab) = (ba)(ba) = (ab)(ba),$$

thus, by quasi-separativity, we have $ab = ba$. So we obtain

$$a^2 = ab = ba = b^2,$$

and by quasi-separativity again we have $a = b$. □

4 Semilattice of a Semigroup

Definition 4.1 (Semilattice)**.** *A* semilattice *is a commutative idempotent semigroup. That is, a set Y with a binary operation*

$$\wedge : Y \times Y \to Y$$

such that, for all $e, f, g \in Y$:

1. $(e \wedge f) \wedge g = e \wedge (f \wedge g)$ *(Associativity),*
2. $e \wedge f = f \wedge e$ *(Commutativity),*
3. $e \wedge e = e$ *(Idempotency).*

Interpretation:

- The operation $\wedge$ can be thought of as a "meet" (greatest lower bound) in an ordered set.
- The partial order associated with a semilattice is given by

$$e \leq f \iff e \wedge f = e.$$

- Thus, a semilattice is both an algebraic structure (a special semigroup) and an order-theoretic one (a meet-semilattice) in a poset.

Definition 4.2 (Semilattice of Semigroups)**.** *Let Y be a semilattice with operation $\wedge$. A* semilattice of semigroups *is a semigroup S together with a decomposition*

$$S = \bigcup_{e \in Y} S_e,$$

where each S_e is a subsemigroup of S, such that for all $e, f \in Y$:

$$S_e \cdot S_f \subseteq S_{e \wedge f}.$$

That is, the product of an element from S_e and an element from S_f always lies in the same component corresponding to the meet $e \wedge f$, see figure 1 below.

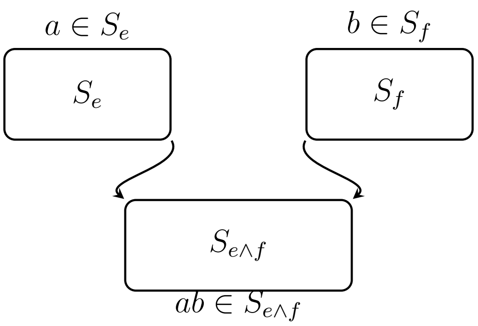

Figure 1: Product in a semilattice of semigroups: $a \in S_e$, $b \in S_f$ multiply into $S_{e \wedge f}$.

Remarks:

- Each component S_e is itself a semigroup.
- The semilattice Y controls how the components interact with each other: multiplication "descends" to the meet in Y.
- If all components S_e are cancellative semigroups, then S is called a *semilattice of cancellative semigroups.*

Definition 4.3 (Green's Relations)**.** *Let S be a semigroup. Green's relations are the equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ defined on S as follows:*

- ***The $\mathcal{L}$-relation:*** *For $a, b \in S$,*

$$a \, \mathcal{L} \, b \quad \Longleftrightarrow \quad S^1 a = S^1 b,$$

that is, a and b generate the same principal left ideal. *Here S^1 denotes S with identity adjoined if necessary.*

- ***The $\mathcal{R}$-relation:*** *For $a, b \in S$,*

$$a \ \mathcal{R} \ b \quad \Longleftrightarrow \quad aS^1 = bS^1,$$

that is, a and b generate the same principal right ideal.

- ***The $\mathcal{J}$-relation:*** *For $a, b \in S$,*

$$a \ \mathcal{J} \ b \quad \Longleftrightarrow \quad S^1 a S^1 = S^1 b S^1,$$

that is, a and b generate the same two-sided ideal.

- ***The $\mathcal{H}$-relation:***

$$\mathcal{H} \ = \ \mathcal{L} \cap \mathcal{R},$$

that is, $a \ \mathcal{H} \ b$ if and only if $a \, \mathcal{L} \, b$ and $a \, \mathcal{R} \, b$.

- ***The $\mathcal{D}$-relation:***

$$\mathcal{D} \ = \ \mathcal{L} \circ \mathcal{R} \ = \ \mathcal{R} \circ \mathcal{L},$$

that is, $a \ \mathcal{D} \ b$ if and only if there exists $c \in S$ such that

$$a \ \mathcal{L} \ c \quad \text{and} \quad c \ \mathcal{R} \ b.$$

Definition 4.4 (Associated Preorders)**.** *Besides the equivalence relations, one often considers the following preorders:*

- *The $\leq_{\mathcal{L}}$-relation: for $a, b \in S$,*

$$a \leq_{\mathcal{L}} b \quad \Longleftrightarrow \quad S^1 a \subseteq S^1 b.$$

- *The $\leq_{\mathcal{R}}$-relation: for $a, b \in S$,*

$$a \leq_{\mathcal{R}} b \quad \Longleftrightarrow \quad aS^1 \subseteq bS^1.$$

- *The $\leq_{\mathcal{J}}$-relation: for $a, b \in S$,*

$$a \leq_{\mathcal{J}} b \iff S^1 a S^1 \subseteq S^1 b S^1.$$

For more clearity, reader should see [3, 8, 10, 17, 19–21]

Theorem 4.1. *Let S be a semigroup. Then $\mathcal{D} = \mathcal{J}$ if and only if for every $a, b \in S$,*

$$SaS = SbS \Rightarrow \exists x \in S \text{ such that } a\,\mathcal{L}\,x\,\mathcal{R}\,b.$$

Proof. ($\Rightarrow$): Suppose $\mathcal{D} = \mathcal{J}$. Let $a, b \in S$ with $SaS = SbS$, i.e. $a\,\mathcal{J}\,b$. Since $\mathcal{D} = \mathcal{J}$, this implies $a\,\mathcal{D}\,b$. By the definition of $\mathcal{D}$, there exists $x \in S$ such that

$$a\,\mathcal{L}\,x \quad \text{and} \quad x\,\mathcal{R}\,b.$$

($\Leftarrow$): Suppose the stated condition holds. Take any $a, b \in S$ with $a\,\mathcal{J}\,b$, i.e. $SaS = SbS$. By hypothesis, there exists $x \in S$ with $a\,\mathcal{L}\,x\,\mathcal{R}\,b$. Hence $a\,\mathcal{D}\,b$.

Since $a, b \in S$ were arbitrary, this shows $\mathcal{J} \subseteq \mathcal{D}$. But in any semigroup it is always true that $\mathcal{D} \subseteq \mathcal{J}$ (see [1, 18, 19]). Therefore, $\mathcal{D} = \mathcal{J}$. □

5 Results

Definition 5.1 (Koch–Wallace Stability [16])**.** *A semigroup S is called* stable *if for all $a, b \in S$:*

1. *(Right stability): $aS \subseteq abS \implies aS = abS$, and*
2. *(Left stability): $Sa \subseteq Sab \implies Sa = Sab$.*

This definition was also given by East using Green's relation in [9].
Equivalently, if $a \leq_{\mathcal{J}} ab$, then $a\mathcal{R}ab$, and if $a \leq_{\mathcal{J}} ba$, then $a\mathcal{L}ba$.

It is now necessary to establish some useful relationship between stable and separative semigroups. This is achieved through the following Propositions and theorems.

Proposition 5.1. *Let $S = \bigcup_{e \in Y} S_e$ be a semilattice (with meet $\wedge$) of cancellative semigroups S_e, so that*

$$S_e S_f \subseteq S_{e \wedge f} \qquad \text{for all } e, f \in Y.$$

Then S is separative.

Proof. Let $a, b \in S$, suppose $a^2 = ab$ and $ba = b^2$.
Let $a \in S_e, b \in S_f$. Since $a^2 \in S_e$ but $ab \in S_{e \wedge f}$, the equality $a^2 = ab$ forces $e = e \wedge f$ and so $e \leq f$.
Likewise, $b^2 \in S_f$ and $ba \in S_{e \wedge f}$ then $b^2 = ba$ forces $f \leq e$.
Hence, $e = f$, and so a, b lie in the same cancellative component S_e.
From $a^2 = ab$, we have $aa = ab \implies a = b$. hus, the first separative implication holds. The second (symmetric) implication is identical in structure. Therefore, S satisfies the separativity conditions and so is separative. □

Theorem 5.1. *Let $S = \bigcup_{e \in Y} S_e$ be a semilattice (index set Y with meet $\wedge$) of cancellative semigroups S_e. If $a \in S_e$ and $b \in S_f$ then $ab \in S_{e \wedge f}$. Then S is stable, i.e. for all $a, b \in S$:*

1. *If* $Sa \subseteq S(ab)$ *then* $Sa = S(ab)$.

2. *If* $aS \subseteq (ab)S$ *then* $aS = (ab)S$.

Proof. Write principal left ideals by $Sa = \{xa : x \in S\}$ and right ideals by $aS = \{ax : x \in S\}$. Let $a \in S_e$ and $b \in S_f$.

(1) Assume $Sa \subseteq S(ab)$. In particular $a \in S(ab)$, so there exists $t \in S$ with $a = t(ab)$. Writing $t \in S_g$, the product $t(ab) \in S_{g\wedge e\wedge f}$, while $a \in S_e$. Equality of components forces $e = g \wedge e \wedge f$, hence $e \leq f$ and so $e \wedge f = e$. Thus $ab \in S_e$.

For any $h \in Y$, the map $\varphi_h : S_{h\wedge e}a \to S_{h\wedge e}(ab)$ defined by $\varphi_h(xa) = (xa)b$ is bijective by cancellativity. Taking unions over h yields $Sa = S(ab)$.

(2) Symmetrically, assume $aS \subseteq (ab)S$. Then $a = (ab)u$ for some $u \in S$. This forces $e \leq f$ and $ab \in S_e$. For each h, the map $\psi_h : aS_{e\wedge h} \to (ab)S_{e\wedge h}$, $\psi_h(ax) = (ab)x$, is bijective. Taking unions gives $aS = (ab)S$.

Hence S is stable. □

Remark 5.1. *The proof uses (i) semilattice decomposition of separative semigroups, and (ii) cancellativity in each component, ensuring injectivity of the maps* $xa \mapsto (xa)b$ *and* $ax \mapsto (ab)x$. *These yield stability (in the sense of [16]).*

Theorem 5.2. *(Burmistrovich's Theorem [10]). A semigroup* S *is separative if and only if it is isomorphic to a semilattice of cancellative semigroups.*

Proof. See [7] □

6 Conclusion

In this work, we have established that every separative semigroup is stable. The argument relies on two fundamental ingredients: first, the structural theorem of Burmistrovich (Theorem 5.2), which asserts that every separative semigroup is isomorphic to a semilattice of cancellative semigroups; second, the fact that semilattices of cancellative semigroups preserve the stability conditions (Theorem 5.1). By combining these facts along with Proposition 5.1, we conclude that the separative property not only encodes a refined form of cancellativity but also guarantees algebraic stability in the sense of Koch and Wallace. This result strengthens the conceptual link between structural decomposition and stability theory in semigroup theory. It shows that separativity, originally introduced to capture subtle cancellation phenomena, naturally enforces stability (in the sense of Koch and Wallace) through its semilattice decomposition. Consequently, the class of separative semigroups provides a robust and stable framework for further exploration in the algebraic theory of semigroups.

Declaration of Interest:

1. Funding: Not applicable

2. Informed Consent Statement: Not applicable

3. Data Availability: Not applicable

4. Conflict of Interest Statement: No conflict of interest.

References

[1] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, *American Mathematical Society* (1961).

[2] B. T. Shourijeh, C^*-Algebras of Some Semigroups, *Honam Mathematical J.* vol. 26, No. 4. pp. 483–507, 2004.

[3] C. Hollings, Mathematics across the Iron Curtain: A History of Soviet Mathematics, *American Mathematical Society*, 2014.

[4] D. D. Miller and Clifford, Regular classes in Semigroups, *Tran. Amer. Math. Soc.* vol. 82, 1956.

[5] D. Rees, On Semi-Groups, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 36, no. 4, pp. 387–400 1940.

[6] E. Hewitt, H. S. Zuckerman, The $L_1\breve{}$algebra of commutative semigroup, *Trans. Amer. Math. Soc.* , 83, pp. 70-97, 1956.

[7] I. E. Burmistrovitch, Commutative bands of cancellative semigroups, *Siberian Mat. Z.* vol. 6, pp. 284–299, 1965.

[8] J.A. Green, On the structure of semigroups, *Ann. of Math.*, pp. 163-–172, 1951.

[9] J. East and P. M. Higgins, Green's relations and stability for subsemigroups, *Semigroup Forum*, vol. 101, No. 1, pp. 77–86, 2020.

[10] O. G. Udoaka, Generators and Inner Automorphism, *The Colloquium-A Multidisciplinary Thematc Policy Journal* vol. 10, No. 1, pp. 102–111, 2022.

[11] L. W. Anderson, R. P. Hunter and R. J. Koch, Some Results on Stability in Semigroups. *AMS Journal*, pp. 521–529, 1965.

[12] M. P. Drazin, A Partial Order in Completely Regular Semigroups, *Journal of Algebra*, vol. 98, pp. 362–374, 1986.

[13] M. Petrich, Introduction to Semigroups, *Charles Babbage Research Centre*, 1984.

[14] N. Kumar and B. Kumar, Some Fundamental Properties of Semigroups and their Classifications, *Internal Journal of Mathematics Trends and Technology*, Vol. 70 Issue 9, pp. 8–11, 2024.

[15] R. Thakur, A Short Note on Completely Regular Semigroup, *International Journal of Creative Research Thoughts* , vol. 11, Issue 5, 2023.

[16] R. J. Koch and A. D. Wallace, Stability in semigroups, *Duke Math. J.*, vol. 24, pp. 193–196, 1957.

[17] R. U. Ndubisi, and O. G. Udoaka, On left Restriction Semigroups, *International Journal of Algebra and Statistics*, vol. 5No. 1, pp. 59–66, 2016. DOI: 10.20454/ijas.1083.

[18] O. G. Udoaka, Rank of some Semigroup, *Int. Journal of Applied Science and Mathematical Theory*, vol. 9, No. 3, pp. 90–100, 2023. www.iiardjournals.org. D.O.I: 10.56201/ijasmt.v9.no3.2023.pg90.100

[19] O. G. Udoaka, O. Tom, and A. Musa, On Idempotent Elements in Quasi-Idempotent Generated Semigroup, *International Journal for Research Trend and Innovation*, vol. 8, No. 11, pp. 2456–3315, 2023.

[20] W. D. Burgess and R. Raphael, On Conrad's partial order relation on semiprime rings and on semigroups, *Semigroup Forum* vol. 16, pp. 133–140, 1978.

[21] X. Mary, On the Structure of Semigroups Whose Regular Elements are Completely Regular. hal-04223812. 2023.

[22] Y. I. Krasilnikova and B. V. Novikove, On quasi-separative Semigroup. arXiv:math/0311414v1 [math.GR], 2003.