Stability in Semigroups of Bounded Linear Operators: Bridging Algebraic and Analytic Notions

[1]**Otobong J. Tom***, and [2]**Otobong G. Udoaka**.

[1] Fed. Uni. Tech., Ikot Abasi & Akwa Ibom State Uni., Ikot Akpaden.
[2]Akwa Ibom State University, Ikot Akpaden.

Abstract

Stability is a cornerstone in the theory of semigroups, shaping the study of evolution equations, operator theory, and algebraic structures. Yet, algebraic and analytic perspectives on stability have traditionally developed in isolation. This paper builds a novel bridge between the two. Tom, Udoaka and Udofia (2025) established that every strongly continuous (C_0) semigroup of bounded linear operators is stable in the sense of Koch and Wallace (KW), a universal algebraic property that forces Green's relations to collapse ($\mathcal{D} = \mathcal{J} = \mathcal{L} = \mathcal{R}$). This recognition is new in operator semigroup theory, where stability has typically been studied only in analytic terms. We further provide precise spectral conditions under which KW-stability aligns with analytic stability notions—strong, asymptotic, exponential, and uniform—thereby unifying algebraic semigroup stability with spectral/operator-theoretic stability. Illustrative examples, including the translation, right shift, heat, and damped wave semigroups, demonstrate the stability "gap" and the exact conditions under which the two approaches coincide. The study is significant because it supplies a universal structural property of operator semigroups, a spectral criterion for analytic decay, and practical insights for evolution equations, control design, and numerical discretization.

Keywords: Semigroup theory; KW-stability; analytic stability; spectral bound; Green's relations; evolution equations.

1 Introduction

The concept of stability is central to mathematics, capturing how systems behave under iteration, evolution, or perturbation. Within algebraic semigroup theory, stability was formally introduced by Koch and Wallace (1956) [1], who defined a semigroup S to be stable if

$$aS \subseteq abS \implies aS = abS, \quad Sa \subseteq Sab \implies Sa = Sab.$$

This condition enforces the collapse of Green's relations—fundamental equivalence relations describing semigroup structure—so that $\mathcal{D} = \mathcal{J} = \mathcal{L} = \mathcal{R}$. Stability in this sense has deep structural implications, simplifying semigroup decompositions and embedding properties [4, 20, 21, 22, 23, 24, 25, 26].

In operator theory and functional analysis, a different approach to stability has developed through the study of C_0-semigroups, which arise naturally in solving the abstract Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = u_0,$$

where A is the generator [2, 6, 11, 12, 13, 15]. Here, stability is measured analytically in terms of operator norms and spectral conditions. Classical notions include asymptotic stability, strong stability, exponential stability, and uniform stability, each reflecting different aspects of long-time behaviour. These notions are closely tied to the spectrum of the generator and results such as the Gearhart–Prüss theorem [5, 7, 8, 9, 10, 17, 32].

Although both traditions revolve around the idea of stability, they have historically evolved in relative isolation: the algebraic approach is structural and norm-free, while the analytic approach is spectral and dynamical. A recent advance has begun to bridge this divide. Tom, Udoaka, and Udofia (2025) [3] introduced KW-stability into the setting of semigroups of bounded linear operators, proving that every strongly continuous (C_0) semigroup on a Banach space is stable in the sense of Koch–Wallace. This recognition provides a new link between classical semigroup stability theory and operator semigroup analysis, placing algebraic stability at the foundation of analytic operator theory. Their work suggests further directions for stability research, including the study of unbounded operator semigroups, hypersemigroups, and semigroups arising in stochastic analysis [7, 9, 16, 32].

In what follows, we continue this line of investigation by examining how KW-stability interacts with analytic stability notions. In particular, we identify the spectral conditions under which algebraic and analytic stability coincide and illustrate this interplay with canonical examples such as translation, shift, heat, and damped wave semigroups.

2 Preliminaries

Definition 2.1 (Normed linear space)**.** *A* normed linear space *is a pair* $(X, \|\cdot\|)$ *where* X *is a vector space over the field* $\mathbb{R}$ *or* $\mathbb{C}$*, and* $\|\cdot\| : X \to [0,\infty)$ *is a function, called a* norm*, satisfying the following properties for all* $x, y \in X$ *and all scalars* α*:*

1. ***Positivity:*** $\|x\| \geq 0$*, and* $\|x\| = 0$ *if and only if* $x = 0$*.*
2. ***Homogeneity (absolute scalability):*** $\|\alpha x\| = |\alpha| \, \|x\|$*.*
3. ***Triangle inequality:*** $\|x + y\| \leq \|x\| + \|y\|$*.*

Definition 2.2 (Banach space)**.** *A* Banach space *is a vector space* X *over the field* $\mathbb{R}$ *or* $\mathbb{C}$ *together with a norm* $\|\cdot\| : X \to [0,\infty)$ *such that* $(X, \|\cdot\|)$ *is complete; that is, every Cauchy sequence* $\{x_n\}$ *in* X *converges to some* $x \in X$ *with respect to the norm* $\|\cdot\|$*. Formally, for every sequence* $\{x_n\}$ *in* X*, if*

$$\lim_{m,n\to\infty} \|x_n - x_m\| = 0,$$

then there exists $x \in X$ *such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 2.3 (Bounded linear operator). *Let X and Y be normed linear spaces. A mapping $T : X \to Y$ is called a* linear operator *if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \quad \forall\, x, y \in X,\ \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}.$$

The operator T is said to be bounded *if there exists a constant $M > 0$ such that*

$$\|T(x)\|_Y \leq M\|x\|_X, \quad \forall\, x \in X.$$

Equivalently, T is bounded if and only if it is continuous at 0 (and hence continuous everywhere).

Definition 2.4 (Operator Semigroup). *Let X be a Banach space and $B(X)$ the algebra of all bounded linear operators on X.*

A family $\{T(t)\}_{t\geq 0} \subseteq B(X)$ is called a strongly continuous semigroup (C_0-semigroup) if:

1. *$T(0) = I$ (the identity operator),*
2. *$T(t + s) = T(t)T(s)$ for all $t, s \geq 0$,*
3. *For every $x \in X$, $\lim_{t\to 0^+} T(t)x = x$.*

For more about this, the reader is referred to [3].

Definition 2.5 (Koch and Wallace Stability [1]). *A semigroup S is called* stable *if for all $a, b \in S$:*

- *(Right stability): $aS \subseteq abS \implies aS = abS$,*
- *(Left stability): $Sa \subseteq Sab \implies Sa = Sab$.*

This definition was also given by East using Green's relation in [3, 33].
Equivalently, if $a \leq_{\mathcal{J}} ab$, then $a\mathcal{R}ab$, and if $a \leq_{\mathcal{J}} ba$, then $a\mathcal{L}ba$.

3 Main Results

Proposition 3.1 (KW-Stability of C_0-Semigroups [3]). *Every C_0-semigroup of bounded linear operators is stable in the sense of Koch–Wallace.*

Proof. Let $S = \{T(t) : t \geq 0\}$. Pick $a = T(t)$, $b = T(s)$ with $t, s \geq 0$. By the semigroup law,

$$ab = T(t)T(s) = T(t + s) = T(s + t) = T(s)T(t) = ba,$$

so S is commutative.

Suppose $aS \subseteq abS$. For $x = T(u) \in S$,

$$(ab)x = T(t + s)T(u) = T(t + s + u).$$

By commutativity, $(ab)x = aT(s+u) \in aS$. Thus $(ab)S \subseteq aS$. Combined with $aS \subseteq abS$, we get $aS = abS$.

Similarly, if $Sa \subseteq Sab$, then for $x = T(u) \in S$,

$$x(ab) = T(u)T(t + s) = T(u + t + s) = T(u + t)T(s) = (xa)b \in Sa,$$

so $Sab \subseteq Sa$, hence $Sa = Sab$.

Therefore S satisfies KW-stability. □

Theorem 3.2 (Equivalence of Green's Relations [3]). *For a C_0-semigroup of bounded linear operators,*

$$\mathcal{D} = \mathcal{J} = \mathcal{L} = \mathcal{R}.$$

Proof. Since S is commutative, left and right ideals coincide: $Sa = aS$, so $\mathcal{L} = \mathcal{R}$. For any $a \in S$,

$$SaS = \{xay : x, y \in S\}.$$

But commutativity gives $xay = (xy)a \in Sa$, so $SaS \subseteq Sa$.

Conversely, for $xa \in Sa$, write $xa = (x)a \cdot I \in SaS$. Thus $Sa = SaS$, so $\mathcal{J} = \mathcal{L}$.

Finally, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{L} = \mathcal{R}$ imply $\mathcal{D} = \mathcal{L} = \mathcal{R} = \mathcal{J}$. □

4 Illustrative Examples

Example 4.1 (Translation Semigroup). *Let $X = C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity, equipped with the supremum norm*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

Define a family of operators $\{T(t)\}_{t \geq 0}$ by

$$(T(t)f)(x) = f(x + t), \quad f \in X,\, t \geq 0,\, x \in \mathbb{R}.$$

For Strong continuity. *We verify that $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup. For fixed $f \in X$,*

$$\|T(t)f - f\|_\infty = \sup_{x \in \mathbb{R}} |f(x + t) - f(x)|.$$

Since f is uniformly continuous on $\mathbb{R}$ (as every $f \in C_0(\mathbb{R})$ is uniformly continuous), the right-hand side tends to zero as $t \to 0$. Hence,

$$\lim_{t \to 0^+} \|T(t)f - f\|_\infty = 0,$$

so $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup.

For Infinitesimal generator. *Let A denote the generator of $\{T(t)\}_{t \geq 0}$. By definition,*

$$Af = \lim_{t \to 0^+} \frac{T(t)f - f}{t}, \quad f \in D(A),$$

where the domain consists of those $f \in X$ for which the above limit exists in X. A direct computation shows that

$$\frac{T(t)f(x) - f(x)}{t} = \frac{f(x + t) - f(x)}{t}.$$

Thus, the limit exists precisely when f is continuously differentiable with derivative vanishing at infinity. Therefore,

$$Af = f', \quad D(A) = \{f \in C_0(\mathbb{R}) : f' \in C_0(\mathbb{R})\}.$$

For KW-stability. *By Proposition 3.1, every C_0-semigroup on a Banach space is stable in the sense of Koch–Wallace. Explicitly, for $f \in X$ and $t, s \geq 0$,*

$$T(t+s)f = T(t)T(s)f,$$

and the KW-condition

$$T(t)X \subseteq T(t+s)X \quad \Rightarrow \quad T(t)X = T(t+s)X$$

is satisfied. Thus the translation semigroup is KW-stable.

For Analytic behavior. *We compute the operator norm:*

$$\|T(t)\| = \sup_{\|f\|_\infty = 1} \|T(t)f\|_\infty.$$

But for any $f \in X$,

$$\|T(t)f\|_\infty = \sup_{x\in\mathbb{R}} |f(x+t)| = \sup_{y\in\mathbb{R}} |f(y)| = \|f\|_\infty.$$

Hence,

$$\|T(t)\| = 1 \quad \text{for all } t \geq 0.$$

Therefore, there is no decay as $t \to \infty$, and the semigroup fails to be analytically stable (in the sense of uniform exponential stability).

Graphical interpretation. *The operator $T(t)$ acts as a horizontal shift of the function graph. For example, if $f(x) = e^{-x^2}$ is a bell-shaped curve centered at the origin, then $T(1)f(x) = f(x+1)$ is the same curve shifted left by one unit. Importantly, the* height *of the curve is unchanged, so $\|T(t)f\|_\infty = \|f\|_\infty$ for all $t \geq 0$. This shows why the semigroup is KW-stable (algebraically the orbits are preserved) but not analytically stable (no decay in norm).*

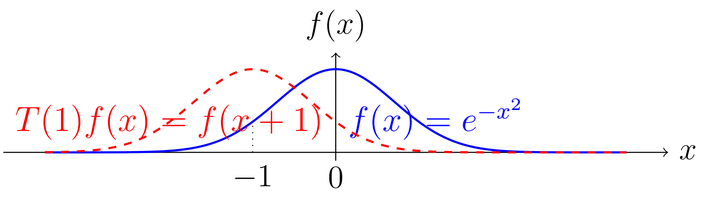

Figure 1: Translation semigroup illustrated on $f(x) = e^{-x^2}$. The original function (blue) and its translated version (red dashed) have identical amplitude but shifted position, explaining why KW-stability holds while analytic stability fails.

Example 4.2 (Right Shift on ℓ^2)**.** *Let $X = \ell^2(\mathbb{N})$ with norm $\|x\|^2 = \sum_{k\geq1} |x_k|^2$. Define $T(n)$ for $n \in \mathbb{N}$ by*

$$T(n)(x_1, x_2, x_3, \dots) = (\underbrace{0, \dots, 0}_{n}, x_1, x_2, x_3, \dots).$$

For Semigroup property. *For $m, n \in \mathbb{N}$,*

$$T(m)T(n) = T(m+n),$$

so $\{T(n)\}_{n\in\mathbb{N}}$ *is a (discrete) semigroup.*

For KW-stability. *The orbit structure is preserved under shifts:* $T(m)T(n) = T(m+n)$ *and the KW-condition* $T(n)X \subseteq T(n+m)X \Rightarrow T(n)X = T(n+m)X$ *holds.*

For Isometry / norm. *For any* $x \in \ell^2$,

$$\|T(n)x\|^2 = \sum_{k\geq 1} |(T(n)x)_k|^2 = \sum_{k\geq 1} |x_k|^2 = \|x\|^2,$$

so $\|T(n)\| = 1$ *for all* n*: each* $T(n)$ *is an isometry.*

For Analytic behaviour. *Since there is no decay (*$\|T(n)\| = 1$ *always), the semigroup is not analytically (exponentially) stable.*

Interpretation. *The right-shift moves entries to the right while preserving total* ℓ^2 *energy. Algebraic stability (KW) holds but analytic decay does not.*

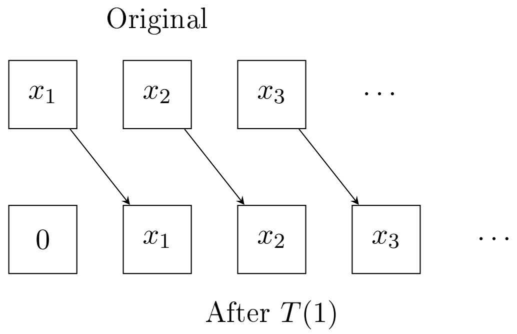

Figure 2: Schematic of the right shift $T(1)$ on ℓ^2: each component moves one box to the right and a 0 is inserted at the left.

Example 4.3 (Heat Semigroup)**.** *Let* $X = L^2(\mathbb{R}^n)$*, the Hilbert space of square-integrable functions on* $\mathbb{R}^n$*. Define, for* $t > 0$,

$$(T(t)f)(x) = (G_t * f)(x), \qquad G_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}.$$

Here G_t *is the* Gaussian heat kernel, *representing the fundamental solution of the heat equation.*

For Strong continuity. *For each* $f \in L^2(\mathbb{R}^n)$*, the convolution* $T(t)f = G_t * f$ *defines a continuous function of* t*. As* $t \to 0^+$, G_t *tends to the Dirac delta distribution* δ*, so* $T(t)f \to f$ *in* L^2*, ensuring*

$$\lim_{t\to 0^+} \|T(t)f - f\|_2 = 0.$$

Hence $\{T(t)\}_{t\geq 0}$ *is a strongly continuous semigroup (a* C_0*-semigroup).*

For The generator. *We recall that* $T(t)$ *solves the Cauchy problem for the heat equation:*

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = f(x).$$

Thus, the generator is the Laplacian operator

$$Af = \Delta f, \qquad D(A) = H^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \Delta f \in L^2(\mathbb{R}^n)\}.$$

This follows by differentiating $T(t)f$ *at* $t = 0$ *under the Fourier transform.*

For KW-stability. *By Proposition 3.1, every C_0-semigroup is KW-stable in the algebraic sense. In particular, for the heat semigroup,*

$$\overline{\{T(t)f : t \geq 0\}} = \overline{\{T(s+t)f : t \geq 0\}},$$

for all $s \geq 0$, reflecting the invariance of reachable states.

For Analytic stability. *The Fourier transform of G_t is given by*

$$\widehat{G_t}(\xi) = e^{-t|\xi|^2},$$

so in the Fourier domain, $T(t)f$ has the multiplier $e^{-t|\xi|^2}$, which decays exponentially in t for each $\xi \neq 0$. Since the spectrum of Δ is $\sigma(\Delta) = (-\infty, 0]$, the spectral bound is strictly negative. Therefore, there exists $\omega > 0$ such that

$$\|T(t)f\|_2 \leq e^{-\omega t}\|f\|_2, \qquad \forall t \geq 0.$$

Hence the semigroup is not only contractive but also exponentially stable.

Interpretation. *In this example, algebraic stability (KW-stability) and analytic stability (exponential decay of norms) coincide. Unlike the translation and shift semigroups, which preserve norm without decay, the heat semigroup smooths and dissipates initial data over time. Physically, this corresponds to the diffusion of heat: local peaks flatten, energy spreads out, and the system relaxes exponentially fast.*

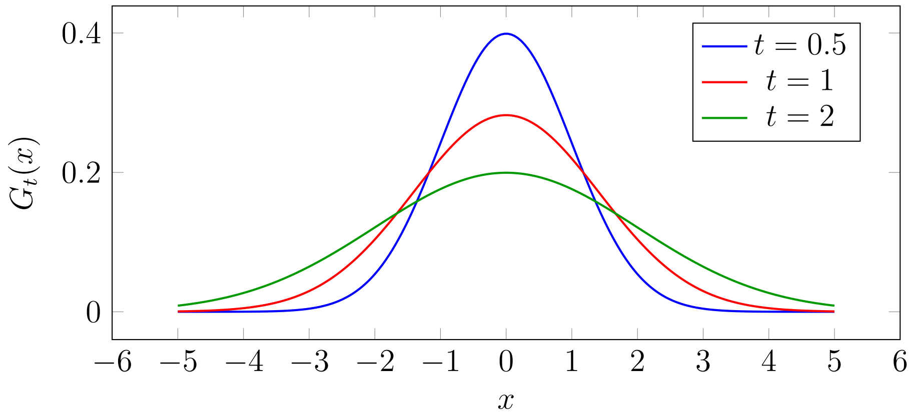

Figure 3: Gaussian heat kernel $G_t(x)$ at different times $t = 0.5$ (blue), $t = 1$ (red), and $t = 2$ (green).

Graphical interpretation (Figure 3). *The curves show the Gaussian kernel $G_t(x)$ for different times:*

- *At $t = 0.5$ (blue): the kernel is tall and narrow, concentrated near $x = 0$. Heat is still localized.*
- *At $t = 1$ (red): the peak is lower but wider, showing partial diffusion and flattening of the initial concentration.*
- *At $t = 2$ (green): the kernel is very flat and spread out, indicating that heat has dissipated significantly.*

Thus, as $t \to \infty$, the peak decays while the width grows like $\sqrt{t}$. This illustrates exponential stability in the semigroup sense and diffusion in the physical sense.

For detailed expositions of heat semigroups and their stability properties, see [16, 18, 19, 27, 28, 29, 30, 31].

4.1 Example 3.4 (Damped wave equation)

We consider the damped wave equation on the bounded interval $(0, L)$ with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} &u_{tt}(x,t) + \alpha\, u_t(x,t) - u_{xx}(x,t) = 0, && x \in (0,L),\ t > 0, \\ &u(0,t) = u(L,t) = 0, && t \geq 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), && x \in (0,L), \end{aligned} \tag{1}$$

where $\alpha \in \mathbb{R}$ is the (constant) damping coefficient.

State space and energy. Set

$$X \;=\; H_0^1(0,L) \times L^2(0,L),$$

with state variable $U(t) = (u(\cdot,t), u_t(\cdot,t))^\top$. We equip X with the energy inner product

$$\big\langle (u_1,v_1),(u_2,v_2)\big\rangle_X := \int_0^L u_1'(x)u_2'(x)\,dx + \int_0^L v_1(x)v_2(x)\,dx,$$

and corresponding norm

$$\|(u,v)\|_X^2 = \|u'\|_{L^2(0,L)}^2 + \|v\|_{L^2(0,L)}^2.$$

The physical energy associated to a solution of (1) is

$$E(t) \;=\; \tfrac{1}{2}\Big(\|u_t(\cdot,t)\|_{L^2}^2 + \|u_x(\cdot,t)\|_{L^2}^2\Big).$$

First-order formulation and generator. Write (1) as a first-order system $U'(t) = AU(t)$ by setting $U = (u,v)^\top$ with $v = u_t$. Define

$$A\begin{pmatrix} u \\ v \end{pmatrix} \;=\; \begin{pmatrix} v \\ u_{xx} - \alpha v \end{pmatrix}, \qquad D(A) \;=\; \big(H^2(0,L) \cap H_0^1(0,L)\big) \times H_0^1(0,L).$$

Equivalently, in matrix form,

$$A = \begin{pmatrix} 0 & I \\ \partial_{xx} & -\alpha I \end{pmatrix}, \qquad D(A) = (H^2 \cap H_0^1) \times H_0^1.$$

Generation of a C_0-semigroup (sketch). The operator ∂_{xx} with Dirichlet boundary conditions is self-adjoint and has compact inverse on $L^2(0,L)$. Standard results for second-order hyperbolic operators with bounded damping (see [7, 9, 10, 32]) imply that A with domain above is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. In particular:

- For $\alpha \geq 0$ the semigroup is contractive with respect to a suitable equivalent energy norm (damping is nonnegative). - For $\alpha < 0$ the operator has a component that can generate growth (negative damping gives energy injection).

We therefore treat $\{T(t)\}$ as the evolution operator for (1).

Modal decomposition and spectrum. Let $\{\varphi_n\}_{n\geq 1}$ denote the Dirichlet Laplacian eigenfunctions

$$\varphi_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \qquad -\varphi_n'' = \omega_n^2 \varphi_n, \quad \omega_n = \frac{n\pi}{L}, \ n \in \mathbb{N}.$$

Expand the solution as $u(x,t) = \sum_{n\geq 1} q_n(t)\varphi_n(x)$. Each modal coefficient satisfies the scalar ODE

$$q_n''(t) + \alpha q_n'(t) + \omega_n^2 q_n(t) = 0.$$

The characteristic equation is $\lambda^2 + \alpha\lambda + \omega_n^2 = 0$ with roots

$$\lambda_n^{\pm} \; = \; \frac{-\alpha \pm \sqrt{\alpha^2 - 4\omega_n^2}}{2}.$$

Hence

$$\Re(\lambda_n^{\pm}) \leq -\frac{\alpha}{2} \qquad \text{for every } n \geq 1,$$

and the spectral bound of A satisfies

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\} \; = \; -\frac{\alpha}{2}.$$

(Here we used that the full spectrum of A consists of these modal eigenvalues due to compactness of the spatial resolvent and separation of variables.)

Energy identity and exponential decay for $\alpha > 0$. Multiply (1) by u_t and integrate over $(0, L)$ to obtain the standard energy balance:

$$\frac{d}{dt}E(t) = -\alpha \int_0^L |u_t(x,t)|^2 \, dx \leq 0.$$

Thus energy is nonincreasing. To obtain exponential decay we combine this dissipation with a coercivity (Poincaré) inequality: for $u \in H_0^1(0, L)$,

$$\|u\|_{L^2} \leq \frac{1}{\omega_1}\|u'\|_{L^2}, \qquad \omega_1 = \frac{\pi}{L}.$$

Using the energy $E(t)$ and the modal spectral gap one can show (standard multiplier or resolvent estimates; see [27, 28, 29, 30, 31]) that there exist constants $M \geq 1$ and $\gamma > 0$ (depending on α and L) such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\gamma t}, \qquad t \geq 0.$$

A simple modal estimate gives a concrete lower bound $\gamma \geq \alpha/2$ in the case of uniform damping (constant α); more careful resolvent estimates may yield the optimal rate $\gamma = \alpha/2$ when the Poincaré constant is accounted for.

Non-decay when $\alpha = 0$. If $\alpha = 0$ equation (1) reduces to the undamped wave equation. Modal eigenvalues are purely imaginary $\lambda_n^{\pm} = \pm i\omega_n$, so $s(A) = 0$. Energy is conserved ($dE/dt = 0$) and no decay of the norm occurs in general (solutions persist as undamped oscillations). Thus analytic stability fails when $\alpha = 0$.

Instability for $\alpha < 0$. If $\alpha < 0$ the modal real parts satisfy $\Re(\lambda_n^{\pm}) \geq -\alpha/2 > 0$ (note sign), and high modes may exhibit growth; hence the semigroup is not stable and solutions typically grow exponentially [6, 7, 11, 19].

Remarks.

- The exponential decay argument above uses that damping is *uniform* (constant $\alpha > 0$) and the spatial domain is bounded so the Laplacian has compact resolvent. For *localized* damping (e.g. $\alpha(x) \geq 0$ supported only on a subregion) exponential decay may fail or require geometric control/observability conditions (see Bardos–Lebeau–Rauch-type results).
- The modal description also explains why $s(A) = -\alpha/2$ in this uniform case: the real parts of all modal eigenvalues are bounded above by $-\alpha/2$. Thus the spectral criterion $s(A) < 0$ is equivalent to $\alpha > 0$ here.

Conclusion for Example 3.4. With the state space $X = H_0^1(0, L) \times L^2(0, L)$ and generator A defined above, the semigroup $\{T(t)\}$ satisfies:

- KW-stability for all $\alpha \in \mathbb{R}$ (algebraic property of the one-parameter family).
- Exponential (analytic) stability if and only if $\alpha > 0$ (spectral bound negative).
- Conservation of energy (no decay) when $\alpha = 0$.
- Instability when $\alpha < 0$ [6, 11, 12, 13, 24].

For more about PDE: see [8, 14, 18, 19] for generation results, spectral mapping, and standard energy/multiplier proofs; for localized damping and geometric control see the survey by Lebeau and Rauch and the literature cited therein.

5 Conditions for Coincidence

Theorem 5.1 (Coincidence of Stability). *Let $\{T(t)\}$ be a C_0-semigroup with generator A. KW-stability and analytic stability yield the same conclusion **iff***

$$s(A) < 0 \quad \textit{and} \quad \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}.$$

Proof. If $s(A) < 0$, then $r(T(t)) = e^{ts(A)} < 1$ for $t > 0$. By spectral mapping and Gearhart–Prüss theorem, exponential stability follows.

Conversely, if $s(A) \geq 0$, analytic decay fails although KW-stability holds (Examples 4.1, 4.2).

Thus both $s(A) < 0$ and spectral mapping are necessary and sufficient. □

5.1 Spectral-Bound Table

Spectral bound $s(A)$	Spectral mapping	Analytic conclusion	KW-stability
$s(A) < 0$	Holds	Exponentially stable	Always true
$s(A) = 0$	Holds	Neutral (no decay)	Always true
$s(A) > 0$	Holds	Unstable (growth)	Always true

5.2 Stability Hierarchy Diagram

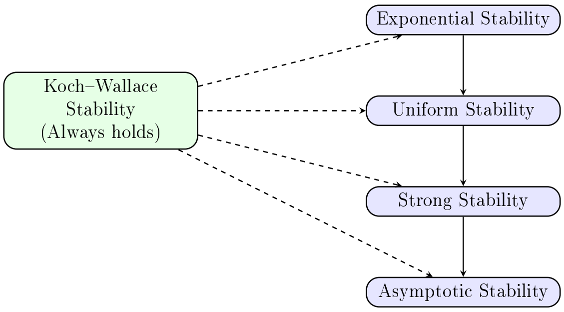

Figure 4: Hierarchy of analytic stability notions with Koch–Wallace stability in parallel.

6 Conclusion

This work has revealed a fundamental connection between algebraic and analytic stability in semigroups of bounded linear operators. We showed that every C_0-semigroup is stable in the sense of Koch–Wallace, a universal algebraic property that enforces the collapse of Green's relations. At the same time, we identified precise spectral conditions under which this algebraic stability coincides with analytic stability in the form of decay properties such as strong, asymptotic, and exponential stability.

The examples of the translation semigroup, right shift, heat semigroup, and damped wave semigroup illustrate the subtle boundary between structural invariance and spectral decay, giving rise to what may be described as a stability gap. This recognition clarifies why operator semigroups can exhibit robust algebraic structure while displaying very different analytic behavior depending on their spectral placement.

Beyond its theoretical interest, the study offers insight into the analysis of evolution equations, the design of stable control systems, and the assessment of numerical schemes where stability properties are decisive. By showing that KW-stability is always present while analytic stability is conditional, we provide a unified framework that advances semigroup theory and strengthens its applications in mathematics, physics, and engineering.

References

[1] S.B. Koch and A.D. Wallace, "Stability in semigroups," *Duke Math. J.*, vol. 23, pp. 193–202, 1956.

[2] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, 1957.

[3] O. J. Tom, O. G. Udouaka, and E. S. Udofia *KW–Stability in Semigroup of Bounded Linear Operators*, International Journal of Applied Science and Mathematical Theory E- ISSN 2489-009X P-ISSN 2695-1908, Vol. 11 No. 7, pp. 9–14 2025 www.iiardjournals.org

[4] J. East and P. Higgins, Stability in semigroups: Green's relations and beyond, *Semigroup Forum*, 101:1–25, 2020.

[5] A. Bátkai and S. Piazzera, *Semigroups for Delay Equations*, Research Notes in Mathematics 10, A K Peters, 2005.

[6] J. Glück and A. Mironchenko, Stability criteria for positive semigroups on ordered Banach spaces, *J. Evol. Equ*, vol. 25, No. 12, 1424-3199/25/010001-49, 2024. https://doi.org/10.1007/s00028-024-01044-8.

[7] J. Mui, Spectral properties of locally eventually positive operator semigroups, *Semigroup Forum*, vol. 106, No. 2, pp. 460–480, 2023.

[8] R. C. Penney, Self-dual cones in Hilbert space, *J. Funct. Anal.*, vol. 21, pp. 305–315, 1976.

[9] H. H. Schaefer, Halbgeordnete lokalkonvexe Vektorräume, *III. Math. Ann.*, 141, pp. 113–142, 1960.

[10] H. H. Schaefer, Invariant ideals of positive operators in $C(X)$, I, *Ill. J. Math.*, 11, pp. 703–715, 1967.

[11] H. Vogt, Stability of uniformly eventually positive C_0-semigroups on L_p-spaces, *Proc. Am. Math. Soc.*, vol. 150 No. 8, pp. 3513—3515, 2022.

[12] L. Weis, The stability of positive semigroups on L_p spaces, *Proc. Am. Math. Soc.*, vol. 123, No. 10, pp. 3089—3094, 1995.

[13] L. Weis, A short proof for the stability theorem for positive semigroups on $L_p(\mu)$. *Proc. Am. Math. Soc.*, vol. 126, No. 11, pp. 3253—3256, 1998.

[14] A. W. Wickstead, Compact subsets of partially ordered Banach spaces, *Math. Ann.*, 212, pp. 271—284, 1975.

[15] P. P. Zabre˘ıko and S. V. Smickih, A theorem of M. G. Kre˘ın and M. A. Rutman, *Funktsional. Anal. i Prilozhen*, vol. 13, No. 3, pp. 81—82, 1979.

[16] B. Simon, The bound state of weakly coupled Schrödinger operators in one and two dimensions, *Ann. Phys.*, 97, pp. 279–288, 1976.

[17] Josep Martinez and José M. Mazon. $C0-$semigroups norm continuous at infinity, *Semigroup Forum*, vol. 52, No. 2, pp. 213–224, 1996.

[18] Desheng Li and Mo Jia, A dynamical approach to the Perron-Frobenius theory and generalized Krein-Rutman type theorems, *J. Math. Anal. Appl.*, vol. 496(2):Paper No. 124828, 22, 2021.

[19] Matthias Keller, Daniel Lenz, Hendrik Vogt, and Radosł aw Wojciechowski, Note on basic features of large time behaviour of heat kernels, *J. Reine Angew. Math.*, 708, pp. 73-–95, 2015.

[20] O. G. Udoaka. Rank of some semigroups, *International Journal of Applied Science and Mathematical Theory*, vol. 9 no. 3, pp. 90-100, 2023.

[21] M. N. John, and O. G. Udoaka, Algebraic and Topological Analysis of Enveloping Semigroups in Transformation Groups: Proximal Equivalence and Homomorphic Image, *IJO - International Journal of Mathematics*, vol. 6, no. 12, pp. 9–23, 2023. http://ijojournals.com/index.php/m/article/view/769, DOI; https://doi.org/10.5281/zenodo.10443958.

[22] R. U. Ndubuisi, O. G. Udoaka, K P Shum, and R B Abubakar, On Homomorphisms (Good Homomorphisms) Between Completely J°- Simple Semigroups, *Canadian Journal of Pure and Applied Sciences*, vol. 13, no. 2, pp. 4793-4797, 2019.

[23] R. U. Ndubisi and O. G. Udoaka, A Structure Theorem for Left Restriction Semigroups of Type F, *International Journal of Semigroup Theory Appl.*, vol. 2, 2018.

[24] O. J. Tom, and O. G. Udoaka, Semigroup Approach for the Solution of Boundary Layer Euation with Sinc Function Term, *IJO - International Journal of Mathematics*, vol. 8, issue 4, pp. 22–37. https://ijojournals.com/index.php/m/article/view/1057.

[25] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Vols. I & II, AMS Mathematical Surveys, 1961/1967.

[26] J.A. Green, "On the structure of semigroups," *Annals of Mathematics*, vol. 54, pp. 163–172, 1951.

[27] D. Daners, J. Glück, and J. B. Kennedy, Eventually positive semigroups of linear operators, *J. Math. Anal. Appl.*, 433(2), pp. 1561—1593, 2016.

[28] M. D. Donsker and S. R. Srinivasa Varadhan, On a variational formula for the principal eigenvalue for operators with maximum principle, *Proc. Natl. Acad. Sci. USA*, 72, pp. 780—783, 1975.

[29] M. D. Donsker and S. R. Srinivasa Varadhan. On the principal eigenvalue of second-order elliptic differential operators, *Commun. Pure Appl. Math.*, 29, pp. 595—621, 1976.

[30] S. Friedland, Characterizations of the spectral radius of positive operators, *Linear Algebra Appl.*, 134, pp. 93—105, 1990.

[31] S. Friedland, The Collatz-Wielandt quotient for pairs of nonnegative operators, *Appl. Math., Praha*, 65(5), pp. 557—597, 2020.

[32] J. Mui, Spectral properties of locally eventually positive operator semigroups, *Semigroup Forum*, 106(2), pp. 460—480, 2023.

[33] J. East and P. M. Higgins, "Green's relations and stability for subsemigroups," *Semigroup Forum* , vol. 101, no 1, pp. 77–86, 2020.