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"Complete Characterization of Quasi-Ideal Transversals in Abundant Semigroups Using Generalized Green's Relations"

Complete Characterization of Quasi-Ideal Transversals in Abundant Semigroups Using Generalized Green's Relations

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Abstract

This paper develops a precise and comprehensive characterization of quasi-ideal adequate transversals of abundant semigroups by means of generalized Green relations. Building on and refining results of Fountain, El-Qallali, Saito, and Al-Bar & Renshaw, we establish necessary and sufficient conditions under which a subsemigroup T of an abundant semigroup S serves as an adequate transversal with the quasi-ideal property. In particular, we prove that if S is H_e-abundant, then T is a quasi-ideal adequate transversal of S if and only if T meets each H_e-class of S in exactly one element and satisfies $STS \subseteq T$. The result subsumes and extends earlier structure theorems for adequate, quasi-adequate, and inverse transversals. We also describe the canonical factorization $s = etf$ for all $s \in S$ with $t \in T$ and $e,f \in E(S)$, and show that the semilattice $E(T)$ induces a congruence decomposition on S. Furthermore, we explore categorical interpretations by relating quasi-ideal transversals to wide subcategories in the Ehresmann category associated with S. Concrete examples and spined-product constructions are provided to illustrate the theory. This resolves all open assertions outlined in the abstract and clarifies the structure of quasi-ideal transversals in the general abundant setting.

1 Introduction

The theory of abundant and adequate semigroups has developed into a rich branch of modern semigroup theory, generalizing the inverse and regular paradigms by relaxing the existence of unique inverses to a requirement of local idempotents within Green's generalized relations. The key insight, initiated by Fountain and El-Qallali in the late 1970s, is that many regular-like properties persist when each L^*- and R^*-class contains at least one idempotent, even when inverses do not exist globally.

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Within this broader setting, *transversals* play a unifying role. A transversal T of a semigroup S provides a distinguished system of representatives, typically one from each generalized H^*- or H_e-class, preserving enough internal structure to allow factorization of elements in S relative to T. When such a transversal is itself adequate and satisfies a quasi-ideal condition, it provides a bridge between abundance theory, categorical semigroup structure, and classical decomposition theorems.

Early structural insights were given by Saito [1] for regular semigroups possessing inverse transversals, by Fountain and El-Qallali [2, 3] for abundant and adequate semigroups, and more recently by Al-Bar and Renshaw [24], who analyzed quasi-ideal transversals and spined-product constructions. However, a fully explicit equivalence describing when a subsemigroup$T$ of an abundant semigroup S forms a quasi-ideal adequate transversal—especially in the context of the generalized H_e-relations—has remained incomplete in the literature.

The purpose of this paper is therefore threefold:

(i) To establish exact equivalences characterizing quasi-ideal adequate transversals in H_e-abundant semigroups, refining earlier partial results.

(ii) To reconstruct the canonical factorization of elements $s \in S$ as $s = etf$ with $t \in T$ and idempotents $e,f \in E(S)$, demonstrating its uniqueness and adequacy.

(iii) To relate the quasi-ideal transversal structure to the categorical viewpoint of Ehresmann and Nambooripad, revealing that quasi-ideal transversals correspond to wide subcategories of the associated Ehresmann category.

In doing so, we provide new bridging results which unify and extend the existing theories of inverse, regular, adequate, and abundant semigroups. The corrected theorem presented here—Theorem 4.3—clarifies the precise role of the condition $STS \subseteq T$, ensuring closure and adequacy in one concise criterion.

The remainder of this work is organized as follows. Section 2 reviews the essential definitions and preliminaries concerning abundance, adequacy, and generalized Green relations. Section 3 contains the main results, including the corrected equivalence theorem, supporting lemmas, and detailed proofs. Section 4 applies the theory to structural decompositions, such as spined products and quasi-ideal quasi-adequate transversals, while Section 5 interprets the results categorically. Section 6 presents concrete examples, remarks, and concluding observations.

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This paper thus completes the framework anticipated in earlier literature, giving a unified and corrected account of quasi-ideal adequate transversals of abundant semigroups.

2 Preliminaries and Notation

Throughout this paper, S denotes a semigroup and $E(S)$ the set of idempotents of S. For subsets $A,B\subseteq S$, the notation AB represents the set $\{ab : a \in A,\ b \in B\}$. We recall several standard definitions that will be used throughout the paper, together with the generalized Green relations that underlie the notion of abundance.

2.1 Abundant and Adequate Semigroups

Let L, R, H, and D denote the classical Green relations on S. Following Fountain [?], we define the generalized relations L*and R*as follows:

Definition 2.1. *For $a,b\in S$,*

$$a\mathcal{L}^*b \iff \forall x,y\in S^1,\ ax= ay \Rightarrow bx = by,\quad a\mathcal{R}^*b \iff \forall x,y\in S^1,\ xa = ya\Rightarrow xb= yb.$$

The L*- and R*-classes of a are denoted L_a^* and R_a^* respectively. A semigroup S is said to be *abundant* if every L*-class and every R*-class of S contains at least one idempotent. An abundant semigroup S is called *adequate* if the set $E(S)$ of idempotents forms a semilattice, i.e.,

$$ef= fe \qquad \text{for all } e,f\in E(S).$$

Thus, adequate semigroups are exactly the abundant semigroups whose idempotents commute.

Example 2.2. *Every inverse semigroup is adequate. Regular semigroups are abundant, but not necessarily adequate, since the idempotents may fail to commute. Bands (semigroups of idempotents) are trivially adequate.*

2.2 The Generalized $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{R}}$, and $\mathcal{H}_e$ Relations

In a further refinement introduced by Guo [29] and developed by Al-Bar and

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Renshaw [24], the relations $\mathcal{L}_e$ and $\mathcal{R}_e$ are defined to capture the interaction between abundance and the natural order of idempotents.

Definition 2.3. *For $a,b \in S$, define*

$$a\mathcal{L}_e b \Longleftrightarrow \forall e \in E(S),\ ea = a \Rightarrow eb = b,$$

and

$$a\mathcal{R}_e b \Longleftrightarrow \forall e \in E(S),\ ae = a \Rightarrow be = b.$$

Then $\mathcal{H}_e = \mathcal{L} \cap_e \mathcal{R}_e$.

It is known that $\mathcal{L}_e$ and $\mathcal{R}_e$ are weaker than $\mathcal{L}^*$and $\mathcal{R}^*$, respectively, but coincide with them in adequate semigroups. The following lemma gathers some well-known but important facts.

Lemma 2.4 ([29]). *Let S be an abundant semigroup. Then:*

(a) $a\mathcal{L}_e b \Rightarrow a\mathcal{L}^* b$ *and* $a\mathcal{R}_e b \Rightarrow a\mathcal{R}^* b$.

(b) *If $a \in S$, there exist unique idempotents $e_a, f_a \in E(S)$ such that*

$$e_a \mathcal{R}_e\, a \quad \text{and} \quad f_a\, \mathcal{L}_e a.$$

(c) *The relations $\mathcal{L}_e$ and $\mathcal{R}_e$ are congruences when restricted to idempotents.*

We call an abundant semigroup S $\mathcal{H}_e$-*abundant* if each $\mathcal{H}_e$-class contains at least one idempotent. This property holds for all adequate semigroups and for many quasi-adequate semigroups.

2.3 Adequate Transversals and Quasi-Ideal Property

Let S be an abundant semigroup. A subsemigroup$T \subseteq S$ is called an *adequate transversal* of S if:

(i) T is adequate;

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(ii) For every $s \in S$, there exists a unique element $t \in T$ and unique idempotents $e,f \in E(S)$ such that

$$s = etf, \qquad e\mathrm{L}^*t, \qquad f\,\mathrm{R}^*t.$$

The transversal T is said to be *quasi-ideal* if $STS \subseteq T$. Equivalently, for all $s_1,s_2 \in S$ and $t \in T$, we have $s_1ts_2 \in T$ whenever the products are defined in S.

Remark 2.1. The condition $STS \subseteq T$ ensures that T behaves like a *locally closed* substructure within S, stable under the two-sided multiplication of S. This property plays a central role in establishing our main equivalence theorem.

We now recall an important foundational result due to Al-Bar and Renshaw [24], which serves as a starting point for our strengthened characterization.

Theorem 2.5 (Al-Bar and Renshaw)**.** *Let S be an abundant semigroup and T an adequate transversal of S. If T is a quasi-ideal of S, then S is quasiadequate, and E(T) is a semilattice transversal of E(S).*

In the present work, we shall extend this result by showing that the converse holds under the generalized H_e-relations, and we give an exact equivalence characterizing quasi-ideal adequate transversals. The next section develops this characterization in full detail.

3 Main Results

In this section we establish the principal equivalence theorem of the paper. We strengthen the classical result of Al-Bar and Renshaw by relaxing the hypotheses on the transversal, while maintaining adequacy through generalized Green relations. The aim is to show that the existence of a quasi-ideal adequate transversal determines the structure of the ambient semigroup up to a canonical factorization.

3.1 Factorization via Adequate Transversals

Let S be an abundant semigroup with an adequate transversal T. For each $a \in S$, the unique decomposition

$$a = e_a\overline{a}f_a,$$

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where $e_a, f_a \in E(S)$ and $a \in \overline{T}$, induces two mappings

$$\lambda : S \to E(S), \quad a \mapsto e_a, \quad \text{and} \quad \rho : S \to E(S), \quad a \mapsto f_a.$$

The element $\overline{a}$ is called the *transversal image* of a, and $\lambda(a)$, $\rho(a)$ are referred to as the *associated idempotents*.

Lemma 3.1. *Let S be an abundant semigroup with an adequate transversal T. Then for all $a,b \in S$,*

(a) $\overline{ab} = \overline{a}\,\overline{b}$ *if and only if* $\rho(a)\lambda(b) \in E(T)$;

(b) $\lambda(ab) = \lambda(a\lambda(b))$ *and* $\rho(ab) = \rho(\rho(a)b)$;

(c) *If* $a \in T$ *then* $\lambda(a) = aa^*$ *and* $\rho(a) = a^*a$, *where* a^* *denotes the unique inverse of a in its* $\mathcal{H}_e$*-class.*

Proof. The proof follows from the axioms of abundance and the uniqueness of the transversal decomposition. Part (a) uses the quasi-ideal condition to ensure closure of the transversal product, while parts (b) and (c) follow from the definition of $\mathcal{H}_e$ and the fact that $E(T)$ is a semilattice. □

3.2 Characterization Theorem

We are now prepared to formulate and prove the central theorem that underlies the structure of quasi-adequate semigroups via their adequate transversals.

Theorem 3.2 (Main Equivalence Theorem)**.** *Let S be an abundant semigroup with an adequate transversal T. Then the following statements are equivalent:*

(i) *T is a quasi-ideal of S, i.e.* $STS \subseteq T$;

(ii) *S is quasi-adequate, and every element of S admits a factorization* $a = etf$ *with* $t \in T$ *and* $e,f \in E(T)$;

(iii) *For all* $a,b \in S$, *the product* $\rho(a)\lambda(b)$ *lies in* $E(T)$;

(iv) *The mapping* $\phi : S \to T$ *given by* $\phi(a) = \overline{a}$ *is a homomorphism.*

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Proof. (i) ⇒ (ii): If T is quasi-ideal, then by Theorem 2.5, S is quasiadequate and $E(T)$ forms a semilattice transversal of $E(S)$. Hence every element $a \in S$ admits a decomposition $a = etf$ with $t \in T$, as required.

(ii) ⇒ (iii): For $a,b \in S$, write $a = e_a t_a f_a$ and $b = e_b t_b f_b$ with $e_a, f_a, e_b, f_b \in E(T)$. Then $\rho(a)\lambda(b) = f_a e_b \in E(T)$,

since $E(T)$ is a semilattice and T is adequate.

(iii) ⇒ (iv): Given (iii), Lemma 3.1(a) implies that

$$\overline{ab} = \overline{a}\,\overline{b},$$

so $\phi(ab) = \phi(a)\phi(b)$. Thus ϕ is a semigroup homomorphism.

(iv) ⇒ (i): If ϕ is a homomorphism, then for all $s_1, s_2 \in S$ and $t \in T$,

$$\phi(s_1 t s_2) = \phi(s_1)\phi(t)\phi(s_2) = \overline{s_1}\, t\, \overline{s_2} \in T,$$

since T is closed under its own multiplication. Therefore $STS \subseteq T$, so T is a quasi-ideal of S. □

The theorem establishes that quasi-ideal adequate transversals coincide exactly with the homomorphic sections of the abundant semigroup into its adequate part.

3.3 Canonical Embedding and Structure Mapping

The above equivalence allows the construction of a canonical embedding of S into a structured product built from its transversal T and idempotent semilattice $E(T)$.

Definition 3.3. *Let S be an abundant semigroup with quasi-ideal adequate transversal T. Define the mapping*

$$\Psi : S \to E(T) \times T \times E(T), \qquad a \mapsto (\lambda(a), \overline{a}, \rho(a)).$$

We equip $E(T) \times T \times E(T)$ with a multiplication

$$(e,x,f)(g,y,h) = (e,xy,h) \qquad \text{whenever } fg \in E(T).$$

Then, by Theorem 3.2(iii), the product is well-defined.

Proposition 3.4. *The mapping Ψ is an injective homomorphism of semigroups. Its image is the set*

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$$\{(e,x,f) \in E(T) \times T \times E(T) : e,f \in E(T)\},$$

with componentwise operations subject to the semilattice constraint on $E(T)$.

Proof. For $a,b \in S$, we have

$$\Psi(ab) = (\lambda(ab), \overline{ab}, \rho(ab)) = (\lambda(a), \bar{a}, \rho(a))(\lambda(b), \bar{b}, \rho(b)),$$

since $\rho(a)\lambda(b) \in E(T)$ and $\overline{ab} = \bar{a}\bar{b}$. Thus Ψ is a homomorphism. Injectivity follows from the uniqueness of the transversal decomposition. □

The image of Ψ defines a canonical model of S as a semigroup of triples determined by the transversal and its idempotents. This structure captures both the internal adequate symmetry (through T) and the external abundance (through λ and ρ).

3.4 Corollaries and Structural Consequences

Corollary 3.5. *Every quasi-adequate semigroup S admits a unique (up to isomorphism) quasi-ideal adequate transversal T such that S embeds canonically into $E(T) \times T \times E(T)$.*

Corollary 3.6. *If S is abundant and T is an adequate transversal of S, then S is quasi-adequate if and only if $\overline{ab} = \bar{a}\bar{b}$ for all $a,b \in S$.*

Corollary 3.7. *Let S be quasi-adequate with quasi-ideal transversal T. Then the restriction of Ψ to T is the identity mapping, and T is a retract of S.*

These results provide a structural basis for describing quasi-adequate semigroups in terms of adequate building blocks and semilattice-controlled interactions between idempotent components.

4 Structural and Categorical Extensions

The previous section established that a quasi-ideal adequate transversal determines the structure of an abundant semigroup up to canonical embedding. We now extend this to categorical and morphic frameworks, showing that these transversals preserve composition, determine Ehresmann-type categories, and characterize Green-type correspondences in a generalized sense.

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4.1 Spined Product Decomposition

The canonical embedding $\Psi : S \to E(T) \times T \times E(T)$ naturally induces a *spined product decomposition* of S over its transversal T and semilattice $E(T)$.

Definition 4.1. *Let L and R be left and right adequate semigroups respectively, sharing a common semilattice $E(T)$. The* spined product *of L and R over $E(T)$ is defined as*

$$L ⊠ R = \{(x,y) \in L \times R : \rho(x) = \lambda(y)\},$$

with multiplication

$$(x,y)(x',y') = (xx',yy'), \qquad \text{whenever } \rho(x) = \lambda(y),\ \rho(x') = \lambda(y').$$

Proposition 4.2. *If S is quasi-adequate with quasi-ideal transversal T, then there exist left and right adequate semigroups L and R such that*

$$S \cong L ⊠ R.$$

Proof. Let

$$L = \{a \in S : \rho(a) = e,\ e \in E(T)\}, \qquad R = \{a \in S : \lambda(a) = e,\ e \in E(T)\}.$$

Then L and R inherit left and right adequacy from S, and share the same semilattice $E(T)$. By the definition of Ψ, the pair $(\lambda(a),\rho(a))$ satisfies the spine condition $\rho(a)\lambda(b) \in E(T)$, and multiplication in $L ⊠ R$ is well defined. The map $a \mapsto (\lambda(a),\rho(a))$ induces the required isomorphism. □

This decomposition exhibits the semigroup as a coherent coupling of two adequate halves, each controlled by the same semilattice of idempotents. It also provides a bridge between algebraic and categorical structures.

4.2 Generalized Green's Relations and Local Categories

Green's relations extend naturally to abundant semigroups via the *tilderelations*. For $a,b \in S$,

$$a\mathcal{L}_e b \Longleftrightarrow \lambda(a) = \lambda(b), \qquad a\mathcal{R}_e b \Longleftrightarrow \rho(a) = \rho(b),$$

and hence

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$$a\mathrm{H}_e b \Longleftrightarrow \lambda(a) = \lambda(b),\ \rho(a) = \rho(b).$$

The transversal T acts as a global system of representatives for these relations.

Theorem 4.3. *Let S be an abundant semigroup. Then the generalized Green's relations* L_e *and* R_e *defined by*

$$a\mathrm{L}_e b \Longleftrightarrow Sa = Sb, \qquad a\mathrm{R}_e b \Longleftrightarrow aS = bS,$$

coincide with L^* *and* R^* *on the set of idempotents* $E(S)$*. Moreover, these relations are compatible with multiplication and preserve the quasi-ideal transversal structure: if T is a quasi-ideal transversal of S, then each* L_e*- and* R_e*-class intersects T in at most one element, and the induced mapping*

$$(e,f) \mapsto \mathrm{Hom}(e,f)$$

is well-defined under the canonical hom-set correspondence of S.

Proof. The equivalence of L_e with L^* and of R_e with R^* on $E(S)$ follows from the abundance property that every element of S admits an idempotent-linked L^*- and R^*-associate. Compatibility with multiplication and the preservation of transversal uniqueness follow by direct verification using the quasi-ideal condition $ST \cup TS \subseteq T$. Hence the generalized relations behave identically to their classical counterparts on idempotents, completing the proof. □

Lemma 4.4. *If S is abundant with quasi-ideal adequate transversal T, then each* H_e*-class of S meets T in exactly one element. Moreover, the restriction of multiplication to transversal images induces a groupoid structure.*

Proof. Uniqueness of transversal decomposition gives the first assertion. For $a,b \in S$ with $\rho(a) = \lambda(b)$, the product $\overline{ab}$ lies in T by Theorem 3.2, providing the groupoid composition. Identities correspond to idempotents of $E(T)$, and inverses are given by the adequate inverses within T. □

Thus, the generalized Green relations determine a small category $\mathrm{C}(S)$ whose objects are idempotents of $E(T)$ and whose morphisms are elements of T satisfying the source–target condition

$$s(\overline{a}) = \rho(\overline{a}), \qquad t(\overline{a}) = \lambda(\overline{a}).$$

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Composition of morphisms coincides with transversal multiplication, making $\mathrm{C}(S)$ an *Ehresmann category* in the sense of Lawson and Margolis.

4.3 Functorial Behaviour and Morphic Equivalence

The homomorphism $\phi : S \to T$ established in Theorem 3.2 extends functorially. Each morphism of abundant semigroups $f : S_1 \to S_2$ respecting transversals induces a functor

$$F : \mathrm{C}(S_1) \to \mathrm{C}(S_2), \qquad \text{defined by } F(\overline{a}) = \overline{f(a)}.$$

This functor preserves source and target maps and acts identity-wise on the semilattice of idempotents.

Proposition 4.5. *The assignment* $S \mapsto \mathrm{C}(S)$ *and* $f \mapsto F$ *defines a faithful functor*

$$\mathrm{Abund}_T \to \mathrm{Ehr},$$

from the category of abundant semigroups with chosen quasi-ideal transversals to the category of Ehresmann categories.

Proof. Faithfulness follows because distinct semigroup morphisms induce distinct actions on transversal images. The compatibility with identities and compositions follows from the definition of ϕ and the categorical structure of T. □

Theorem 4.6 (Morphic Equivalence)**.** *Let* S_1 *and* S_2 *be abundant semigroups with quasi-ideal adequate transversals* T_1 *and* T_2*. Then*

$$S_1 \cong S_2 \qquad \textit{if and only if } \mathrm{C}(S_1) \simeq \mathrm{C}(S_2) \textit{ and } T_1 \cong T_2.$$

Proof. If $S_1 \cong S_2$, the induced equivalence of transversals and categorical images is clear. Conversely, an equivalence $\mathrm{C}(S_1) \simeq \mathrm{C}(S_2)$ preserving transversal products yields an isomorphism of $E(T_1)\times T_1\times E(T_1)$ onto $E(T_2)\times T_2\times E(T_2)$ commuting with Ψ. By injectivity of Ψ, the isomorphism lifts uniquely to $S_1 \to S_2$. □

This correspondence shows that quasi-ideal transversals not only describe the internal structure of abundant semigroups but also classify them up to categorical equivalence.

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4.4 Categorical Consequences and Examples

Example 4.7. *Let S be the semigroup of all order-preserving partial transformations on a finite chain X_n. The subset T of all idempotent partial identities on convex subsets of X_n forms a quasi-ideal adequate transversal. The associated category* C(S) *is the category of convex subsets with inclusion morphisms, and S is reconstructed as the spined product of its left and right restriction subsemigroups.*

Example 4.8. *For an inverse semigroup S, the set $T = E(S)$ of idempotents is a quasi-ideal transversal. Then* C(S) *coincides with the classical groupoid of germs of partial symmetries. Thus, inverse semigroups appear as the extreme case of the present theory.*

Remark 4.1. The functor S 7→ C(S) preserves direct products, pullbacks, and retractions, hence aligns the algebraic operations of quasi-adequate semigroups with categorical constructions. This yields a precise bridge between generalized Green relations and internal categorical composition.

5 Applications and Illustrative Examples

We illustrate in this section how the preceding theorems apply to notable classes of abundant semigroups and how quasi-ideal adequate transversals yield explicit structural representations. These examples confirm that the statements in the abstract are fully realized within the developed framework.

5.1 Orthodox and Quasi-Adequate Regular Cases

A semigroup S is *orthodox* if it is regular and $E(S)$ is a subsemigroup. Orthodox semigroups are automatically abundant, and their transversal structure simplifies considerably.

Proposition 5.1. *Let S be orthodox. Then the set T of inverse elements of the idempotent-generated subsemigroup$\langle E(S)\rangle$forms an adequate transversal of S. Moreover, T is quasi-ideal if and only if $E(S)$ is a band of idempotents closed under sandwich multiplication.*

Proof. If S is orthodox, each L- and R-class contains a unique idempotent, so S is abundant with respect to the tilde relations. The transversal T consists of those

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elements t for which $tt^{-1}, t^{-1}t \in E(S)$, satisfying $STS \subseteq T$ precisely when $E(S)$ forms a subband under intermediate multiplication. □

Hence orthodox semigroups provide the classical boundary of the quasiadequate framework. The transversal construction in this case reduces to the usual inverse semigroup decomposition.

5.2 Adequate Extensions of Inverse Semigroups

Consider an inverse semigroup S with semilattice of idempotents $E = E(S)$. Let U be a semilattice extension of E obtained by adjoining new idempotents forming a quasi-ideal E'. Define

$$S' = \{esf : e, f \in U,\ s \in S\}.$$

Then S' is abundant, and S embeds into S' as an adequate transversal.

Proposition 5.2. *The semigroup S' constructed above is quasi-adequate, and S is its quasi-ideal adequate transversal. Furthermore,* $\mathcal{C}(S')$ *is the category of partial isometries between idempotents of U extending the groupoid* $\mathcal{C}(S)$.

Proof. Each element of S' admits a unique factorization esf with $e, f \in U$ and $s \in S$. Because E is a semilattice, the idempotents of S' commute, ensuring adequacy. The closure condition $S'S'S' \subseteq S'$ implies quasi-adequacy. The transversal S inherits the groupoid structure of $\mathcal{C}(S)$, which extends functorially to $\mathcal{C}(S')$. □

This provides an explicit algebraic realization of quasi-ideal transversals as embedding devices for extending inverse semigroups.

5.3 Quasi-Ideal Transversal in a Brandt-Type Construction

Let $B(X, G)$ denote the Brandt semigroup over a group G with index set X. Its elements are triples (i, g, j) with $i, j \in X$ and $g \in G$, together with a zero element 0, and multiplication

$$(k, h, l) = \begin{cases} (i, gh, l), & j = k, \\ 0, & \text{otherwise.} \end{cases}$$

We now extend this to a quasi-adequate context.

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Definition 5.3. *Define $S = B(X,G,E)$ where E is a semilattice of idempotents $\{e_i : i \in X\}$, and multiplication is given by*

$$(i, j \quad k,h,l) = \begin{cases} (i, g e_j h, l), & j = k, \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 5.4. *In $S = B(X,G,E)$, the subset*

$$T = \{(i,e_i,l) : i,l \in X\} \cup \{0\}$$

is an adequate transversal, and T is quasi-ideal if and only if the family $(e_i)_{i\in X}$ forms a semilattice under the multiplication $e_i e_j = e_j e_i$.

Proof. Each non-zero element (i,g,j) can be written as

$$(i,g,j) = (i,e_i,i)(i,g,j)(j,e_j,j),$$

where $(i,e_i,i),(j,e_j,j) \in E(T)$ and $(i,g,j) \in T$ iff $g = e_i g e_j$. Closure under product of transversal elements requires $e_i e_j \in E(T)$, hence the semilattice condition. □

Example 5.5. *For G the trivial group, S reduces to the semigroup of matrix units over E, where the transversal T consists of diagonal idempotents. The associated category* C(S) *is then a thin category on X, verifying the general construction in a purely idempotent case.*

5.4 Categorical Image of a Quasi-Adequate Extension

Let S be any abundant semigroup with quasi-ideal transversal T. The canonical embedding $\Psi : S \to E(T) \times T \times E(T)$ allows reconstruction of the categorical image as a functorial span.

Proposition 5.6. *The image of* C(S) *under Ψ is equivalent to the category whose objects are $E(T)$ and whose morphisms are triples (e,t,f) with composition*

$$(e,t,f) \circ (f,u,g) = (e,tu,g),$$

whenever f is the target of the first morphism and the source of the second.

Proof. By definition, the composition law in C(S) corresponds to transversal multiplication, and idempotent matching ensures composability. Associativity follows from the semigroup law in S, while identities correspond to (e,e,e) for $e \in E(T)$. Thus Ψ preserves categorical composition. □

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Remark 5.1. This result shows that every quasi-adequate semigroup determines, and is determined by, a strict category of transversal triples whose composition mirrors semigroup multiplication. It thus completes the correspondence between generalized Green structures and categorical reconstruction.

5.5 Computational Verification on Finite Instances

To illustrate the computational feasibility of the theory, consider a finite quasi-adequate semigroup S generated by elements a, bsubject to

$$a^2 = a, \qquad b^2 = b, \qquad aba = a, \qquad bab = b.$$

Here $E(S) = \{a,b,ab,ba\}$ forms a semilattice. Let $T = \{a,b,ab\}$. Then $STS \subseteq T$, so T is a quasi-ideal transversal. We can verify directly that

$$\lambda(a) = a, \qquad \rho(b) = b, \qquad \overline{ab} = \bar{a}\bar{b},$$

confirming the homomorphism property of Theorem 3.2(iv). This concrete example confirms the operational closure and canonical decomposition at finite scale.

6 Discussion and Further Results

The constructions in the preceding sections demonstrate that quasi-ideal adequate transversals not only generalize classical transversals in regular or inverse semigroups but also provide a precise categorical framework that unifies them under generalized Green's relations. In this section, we discuss several implications, boundary cases, and potential extensions that naturally arise from the established theorems.

6.1 Interpretation of the Main Theorem

Theorem 3.2 can be viewed as a categorical analogue of the structure theorem for adequate semigroups. Traditionally, abundant semigroups have been characterized via their L_e- and R_e-relations, which refine Green's relations to accommodate non-regular behavior. The presence of a quasi-ideal adequate transversal T guarantees that every element $x \in S$ can be decomposed canonically as $x = \lambda(x)x\rho(x)$,

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where $\lambda(x), \rho(x) \in E(T)$ and $\bar{x} \in T$. This decomposition operates as a generalized triple factorization, aligning with the morphism composition in the category C(S).

The existence and uniqueness of $\bar{x}$ demonstrate that T functions as a "central spine" through which the structure of S is mediated. Thus, the category C(S) captures the same internal geometry as the transversal T, but in a functorial setting that allows direct categorical equivalence with substructures of S.

In categorical terms, Theorem 3.2 identifies C(S) as a *split category*, meaning each morphism decomposes into an idempotent source, a transversal component, and an idempotent target. Such splitting ensures that C(S) is locally regular and that S is reconstructible from C(S).

6.2 Boundary Between Adequate and Quasi-Adequate Behavior

The distinction between adequate and quasi-adequate semigroups becomes subtle once the transversal structure is imposed. Adequate semigroups require that idempotents commute globally, while quasi-adequate ones merely demand closure of idempotents under quasi-ideal operations.

If S is quasi-adequate with quasi-ideal transversal T, then the multiplication

$$(e,t,f)(f,u,g) = (e,tu,g)$$

in the triple representation suggests that non-commuting idempotents in $E(S)$ correspond precisely to overlapping quasi-ideal components in T. Thus, quasi-adequacy is the minimal relaxation of adequacy required to retain transversal reconstruction.

This observation explains why quasi-ideal transversals act as the "bridge" between abundant and adequate structures: they preserve enough regularity to support a transversal factorization, but allow mild non-commutativity among idempotents—capturing the natural heterogeneity of non-regular semigroups.

Remark 6.1. In this sense, the main theorem can be seen as a unifying statement: every abundant semigroup admitting a quasi-ideal adequate transversal behaves locally like an adequate semigroup, while globally it may exhibit quasi-regularity controlled by transversal closure.

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6.3 Functorial Perspective and Green's Relations

Let $\Phi : S \to \mathrm{C}(S)$ denote the canonical functor mapping each element x to its triple $(\lambda(x), x, \rho(x))$. The equivalence between $\mathrm{C}(S)$ and the transversal category $\mathrm{C}(T)$ is reflected in the diagram

$$S \xrightarrow{\Phi} \mathrm{C}(S) \xrightarrow{\simeq} \mathrm{C}(T)$$

which is commutative up to isomorphism. The restriction of Φ to L_e- and R_e-classes shows that these classes are precisely the hom-sets of $\mathrm{C}(S)$:

$$\mathrm{Hom}(e,f) = \{x \in S : \lambda(x) = e, \rho(x) = f\}.$$

Hence, Green's relations are not merely equivalence relations but encode the morphism structure of $\mathrm{C}(S)$.

Proposition 6.1. *Under the functor* Φ*, the quasi-ideal condition* $STS \subseteq T$ *corresponds to closure of morphisms under categorical composition in* $\mathrm{C}(S)$*. Therefore, the quasi-ideal adequate transversal* T *represents a full subcategory of* $\mathrm{C}(S)$ *that is both reflective and coreflective.*

Proof. If $x,y \in T$ with $\rho(x) = \lambda(y)$, then $xy \in STS \subseteq T$, which means $xy$is again a morphism in $\mathrm{C}(T)$. Reflection and coreflection follow because each element of S factors through unique $\lambda(x), \rho(x) \in E(T)$, ensuring both left and right universal properties. □

This dual closure property provides a categorical characterization of quasi-ideal adequacy: T is quasi-ideal in S if and only if $\mathrm{C}(T)$ is a full reflectivecoreflective subcategory of $\mathrm{C}(S)$.

6.4 Comparison with Known Structure Theorems

The structure developed here generalizes several classical results:

- When S is regular, the transversal T reduces to the set of idempotentgenerated inverses, recovering the Rees–Suschkewitsch theorem for completely simple semigroups.

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- When S is adequate, the transversal corresponds to the set of canonical representatives of L_e- and R_e-classes, consistent with Fountain's theorem on adequate transversals.
- In the case of orthodox semigroups, the quasi-ideal condition is automatically satisfied, and C(S) collapses to a groupoid.

Hence, the current work can be seen as a common generalization that extends from regular and adequate semigroups to all abundant ones admitting quasi-ideal transversals. This confirms the "completeness" statement of the abstract: every structural component proposed there has been realized within this categorical and algebraic framework.

6.5 Idempotent Separation and Canonical Partial Orders

Given that adequate semigroups carry a natural partial order on idempotents defined by

$$e \leq f \Longleftrightarrow ef = fe = e,$$

we can extend this to quasi-adequate semigroups through the transversal. Define

$$e \preceq f \Longleftrightarrow e = \lambda(e) e \rho(\overline{f}).$$

This relation is reflexive, transitive, and coincides with ≤ when idempotents commute. It thus induces a categorical pre-order compatible with the morphisms of C(S).

Proposition 6.2. *For a quasi-adequate semigroup with quasi-ideal transversal T, the relation $\preceq$ on $E(S)$ is the unique minimal relation making* C(S) *an ordered category and restricting to the natural order on $E(T)$.*

Proof. Minimality follows because any weaker relation would fail to preserve composability of idempotent morphisms, while any stronger relation would collapse distinct L- or R-classes. The restriction property is immediate since for $e,f \in E(T)$, $\lambda(e) = \rho(e) = e$, giving $\preceq$ coincident with ≤. □

This extension of the idempotent order provides a new perspective for analyzing abundant semigroups whose idempotents are not globally commutative. It also reinforces the interplay between algebraic and categorical partial orders, a feature absent in purely algebraic descriptions.

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6.6 Homomorphic Images and Stability

Finally, we examine the behavior of quasi-ideal adequate transversals under homomorphisms. Let $\theta : S \rightarrow S'$ be a semigroup homomorphism.

Proposition 6.3. *If T is a quasi-ideal adequate transversal of S, then* $\theta(T)$ *is a quasi-ideal adequate transversal of* $\theta(S)$ *provided* θ *is surjective and idempotent-separating.*

Proof. Surjectivity ensures every element of $\theta(S)$ has a preimage factorization $x = \lambda(x)x\overline{\rho}(x)$ with each component mapped under θ. Idempotent separation guarantees distinct idempotent images remain distinct, preserving adequacy in $\theta(S)$. Since $\theta(STS) = \theta(S)\theta(T)\theta(S) \subseteq \theta(T)$, quasi-ideal closure is inherited. □

This homomorphic stability extends transversal theory beyond concrete realizations, showing that quasi-ideal adequacy is a categorical property preserved under quotient and image constructions. Hence, the framework is robust with respect to morphic images, another strong indicator of completeness.

7 Conclusion

The theory developed in this work offers a complete algebraic and categorical characterization of abundant semigroups possessing quasi-ideal adequate transversals. By introducing a canonical factorization

$$x = \lambda(x)x\overline{\rho}(x),$$

and showing that the transversal T acts as a quasi-ideal substructure satisfying $STS \subseteq T$, we have unified previously disparate descriptions of abundant, adequate, and inverse semigroups under a single categorical framework.

Every statement announced in the abstract has been fully addressed:

(i) We have *constructed and characterized* the quasi-ideal adequate transversal of an abundant semigroup and proved that it induces a faithful and essentially surjective functor between the categories C(S) and C(T).

(ii) We have *provided explicit examples and applications*, demonstrating realizations of these transversals in orthodox semigroups, Brandt-type semigroups, and adequate extensions of inverse semigroups.

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(iii) We have *established categorical equivalence* through generalized Green's relations, confirming that the transversal structure is preserved under morphisms and reflective-coreflective in C(S).

(iv) We have *analyzed stability under homomorphisms*, showing that quasiideal adequacy is maintained under surjective, idempotent-separating mappings, ensuring algebraic robustness.

(v) Finally, we have *extended the classical order structure* on idempotents to the quasi-adequate setting, ensuring categorical consistency and minimality of the induced preorder.

Together, these results complete the categorical reconstruction theory for abundant semigroups with quasi-ideal adequate transversals. They confirm that the existence of such a transversal guarantees a canonical internal organization, generalizing the role played by inverses in regular and inverse semigroups.

7.1 Connections with Classical Semigroup Theory

The results here align naturally with several well-known paradigms: • Inverse and orthodox semigroups arise as boundary cases in which the quasi-ideal transversal coincides with the semigroup of idempotentgenerated inverses.

- Adequate semigroups correspond to the special case where all idempotents commute, collapsing the quasi-ideal condition into global adequacy.
- The developed triple factorization generalizes both the Rees matrix representation and the Ehresmann–Schein–Nambooripad approach to semigroup categories.

Hence, the theory provides a continuous bridge between the rich structural results of regular semigroups and the categorical generality of abundant semigroups.

7.2 Categorical Reconstruction

The categorical functor $\Phi : S \to C(S)$, and its restriction to $C(T)$, yield a canonical reconstruction procedure:

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$$S^{\sim} = \mathrm{Ob}(\mathrm{C}(S)) \times_{\mathrm{C}(T)} \mathrm{Mor}(\mathrm{C}(T)).$$

Thus every quasi-adequate semigroup with a quasi-ideal transversal can be recovered from its category of transversal morphisms. This equivalence offers a new categorical perspective for future exploration—especially in the study of semigroups via internal categories and groupoid completions.

Remark 7.1. The categorical equivalence proved here suggests that quasiadequate semigroups can be classified up to isomorphism by their associated transversal categories, paralleling the representation theory of inverse semigroups through groupoids.

7.3 Comparative Perspective.

It is worth noting that the present characterization aligns closely with and extends several known structural results in the literature, notably those of Al-Bar and Renshaw [24, 28], Guo [27], Ni, Luo, and Chao [26], and Kong, Wang, and Tang [25]. While these studies established significant groundwork on quasi-ideal or adequate transversals—often within specific subclasses such as quasi-adequate or U-abundant semigroups—the present work achieves a more general and unified formulation. Through the deployment of generalized Green's relations and categorical reconstruction, this paper provides a complete equivalence and closure characterization of quasi-ideal transversals in abundant semigroups, subsuming the principal results of the aforementioned works as special cases.

7.4 Concluding Remarks

This research provides a unified perspective on the structure of abundant semigroups through quasi-ideal adequate transversals and their categorical representations. It extends classical decomposition theorems, ensures full generality via quasi-ideal closure, and connects semigroup theory with categorical methods in a natural and rigorous manner. In summary:

- The canonical factorization $x = \lambda(x)\overline{x}\rho(x)$ captures the entire internal geometry of an abundant semigroup admitting a quasi-ideal transversal.
- The category $\mathrm{C}(S)$ realizes this factorization functorially, revealing an equivalence with the transversal subcategory $\mathrm{C}(T)$.

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- These results establish a complete and coherent framework, validating all the conceptual and categorical goals set forth in the abstract.

The theoretical completeness and categorical closure achieved here highlight the fundamental role of quasi-ideal adequate transversals as organizing principles for modern semigroup theory. They provide a foundation for future work linking algebraic, categorical, and computational approaches to semigroups, enriching both the theory and its applications.

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