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"A Structural and Categorical Study of Quasi-Ideal Transversals in Abundant Semigroups"

A Structural and Categorical Study of Quasi-Ideal Transversals in Abundant Semigroups

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Abstract

Quasi-ideal transversals serve as central tools for analyzing the structure of abundant semigroups. While generalized Green's relations—L ∗ , R∗ , and H∗—extend classical decomposition techniques, a full characterization of quasi-ideal transversals in terms of these relations has remained incomplete. Existing studies are mostly restricted to adequate or special U-abundant semigroups, lacking a unified structural and categorical framework. This paper develops a comprehensive theory of quasi-ideal transversals in abundant semigroups using generalized Green's relations. We provide necessary and sufficient conditions for their existence formulated purely through L ∗ - and R∗ -classes, and prove canonical factorization theorems expressing every element as a "sandwich" of transversal representatives and related elements. Building on these factorizations, we establish categorical reconstruction results, showing that an abundant semigroup can be recovered up to isomorphism from transversal data and generalized Green's classes. Compatibility with morphisms and closure under sandwich operations is also demonstrated, integrating and extending previous results on adequate and quasi-adequate transversals. Our findings offer a unified structural and categorical treatment of quasi-ideal transversals, consolidating disparate results in the

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literature and providing a foundation for further algebraic and categorical investigations in non-regular semigroup theory.

Introduction

Green's relations have played a central role in semigroup theory since their introduction by Green [3], providing a powerful means of analyzing the internal structure of semigroups through equivalence classes determined by principal ideals. In the regular and inverse settings, these relations lead to precise structural decompositions and representation results, many of which are now classical. The theory of semigroups has been extensively studied through the lens of Green's relations [3, 1, 2], which allow a detailed understanding of internal structure and class decomposition. In non-regular settings, generalized Green's relations, including L^*, R^*, and H^*, have been used to extend classical structural insights to abundant and U-abundant semigroups [4, 14, 16]. A comprehensive characterization of quasi-ideal adequate transversals in abundant semigroups via generalized Green's relations has been recently established, providing canonical decompositions and categorical interpretations for these structures [10].

Transversals, and particularly quasi-ideal transversals, provide a powerful mechanism for decomposing semigroups into canonical components [5, 19, 13]. These tools have been successfully applied to structural analysis, canonical factorization, and categorical reconstruction [15, 12].

Recent work has also highlighted intrinsic limitations of classical structural characterizations based on independence and generating systems in semigroups, demonstrating that such notions do not always capture the internal organization of non-regular and non-commutative semigroups adequately [11]. This reinforces the need for alternative structural frameworks—such as generalized Green's relations and transversal-based decompositions—capable of encoding deeper algebraic behavior.

In non-regular contexts, however, the classical Green's relations are often too coarse to capture essential structural features. This limitation motivated the introduction of generalized Green's relations, notably the relations L^*, R^*, and H^*, which arise naturally in the study of abundant semigroups following the work of Fountain [4]. These relations refine the classical framework while retaining sufficient flexibility to apply beyond regular semigroups.

Parallel to these developments, the concept of a transversal has emerged as a key tool for decomposing semigroups into more manageable components. In particular, adequate and quasi-ideal transversals have been used to isolate canonical representatives from Green-type classes and to facilitate reconstruction of the ambient semigroup. Foundational work by El-Qallali and Fountain [5] established the importance of transversals in abundant semigroups, while later contributions refined the theory in special settings.

Despite these advances, existing results concerning quasi-ideal transversals remain fragmented. Studies such as those of Al-Bar and Renshaw [6] and Kong, Wang, and Tang [7] focus primarily on specific subclasses of abundant semigroups or emphasize structural decomposition without establishing complete equivalence results formulated purely in terms of generalized Green's relations. Moreover, the categorical implications of quasi-ideal transversals—particularly their role in reconstructing semigroups up to isomorphism—have not been fully developed.

The purpose of the present paper is to provide a comprehensive and unified theory of quasi-ideal transversals in abundant semigroups using generalized Green's relations. We establish necessary and sufficient conditions for the existence of such transversals, prove canonical factorization theorems, and develop categorical reconstruction results that extend and consolidate earlier work.

The paper is organized as follows. Section 2 recalls the necessary preliminaries on semigroups, abundant structures, and generalized Green's relations. In Section 3 we examine structural properties of abundant semigroups relevant to transversal theory. Section 4 introduces quasi-ideal transversals and studies their basic properties. The main equivalence theorems are proved in Section 5, followed by canonical factorization results in Section 6. Section 7 develops categorical reconstruction results, while Section 8 investigates closure properties under standard operations. Examples and applications are presented in Section 9, and the paper concludes with a summary and directions for future research.

1 Preliminaries and Definitions

Throughout this paper, all semigroups are assumed to be associative and written multiplicatively. Standard terminology follows Clifford and Preston [1] and Howie [2] unless stated otherwise.

1.1 Semigroups and Idempotents

Definition 2.1. A *semigroup* is a nonempty set S equipped with an associative binary operation.

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Definition 2.2. An element $e \in S$ is called an *idempotent* if $e^2 = e$. The set of all idempotents of S is denoted by $E(S)$.

Idempotents play a fundamental role in the structure theory of non-regular semigroups, particularly in the abundant setting.

1.2 Green's Relations

For $a,b \in S$, the classical Green's relations L, R, and H are defined by

$$a\mathcal{L}b \Longleftrightarrow S^1a = S^1b, \qquad a\mathcal{R}b \Longleftrightarrow aS^1 = bS^1,$$

where S^1 denotes S with an identity adjoined if necessary, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

1.3 Abundant Semigroups

Definition 2.3. A semigroup S is called *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains at least one idempotent.

Abundant semigroups generalize regular and inverse semigroups while retaining sufficient structural control via idempotents. For foundational definitions of abundant semigroups and their properties, see [4, 8, 17]. Canonical factorization results in this context build upon the work of Li [15] and classical transversal theory [5, 19].

1.4 Generalized Green's Relations

We now recall the generalized Green's relations introduced by Fountain [4].

Definition 2.4. For $a,b \in S$, define

$$a\mathcal{L}^* b \Longleftrightarrow (\forall x,y \in S^1)\ xa = ya \Longleftrightarrow xb = yb,$$

and dually,

$$a\mathcal{R}^* b \Longleftrightarrow (\forall x,y \in S^1)\ ax = ay \Longleftrightarrow bx = by.$$

Let $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$.

Remark 2.5. In regular semigroups, $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$, so the generalized relations genuinely extend the classical ones.

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These structural properties have been explored in detail in the literature [4, 8, 16, 14]. The behavior of idempotents and H*-classes in abundant semigroups forms the basis for canonical factorization and categorical reconstruction [15, 12].

1.5 Transversals and Quasi-Ideals

Definition 2.6. Let S be a semigroup. A subset $T \subseteq S$ is called a *transversal* of a family of equivalence classes if T intersects each class in exactly one element.

Definition 2.7. A subset $Q \subseteq S$ is called a *quasi-ideal* of S if

$$QSQ \subseteq Q.$$

Definition 2.8. Let S be an abundant semigroup. A subset $T \subseteq S$ is called a *quasi-ideal transversal* if:

(i) T is a transversal of the H*-classes of S; (ii) T

is a quasi-ideal of S.

This notion generalizes adequate transversals and will serve as the central object of study in the sequel.

The quasi-ideal property, $TST \subseteq T$, ensures closure under sandwich operations and under canonical factorization [19, 13]. Examples and constructions of quasi-ideal transversals in computational settings can be found in [15, 17].

2 Structural Properties of Abundant Semigroups

This section develops structural properties of abundant semigroups that are essential for the analysis of quasi-ideal transversals. Particular emphasis is placed on the behavior of idempotents within generalized Green's classes and on canonical representatives arising from L*- and R*-relations.

2.1 Idempotents in L*- and R*-Classes

A defining feature of abundant semigroups is the guaranteed presence of idempotents in generalized Green's classes. This property underpins most structural decomposition results.

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Lemma 3.1. *Let S be an abundant semigroup and let $a \in S$. Then there exist idempotents $e, f \in E(S)$ such that*

$$e\mathcal{L}^* a \qquad \text{and} \qquad a\mathcal{R}^* f.$$

Proof. By definition of abundance, every $\mathcal{L}^*$-class contains at least one idempotent, so there exists $e \in E(S)$ with $e\mathcal{L}^* a$. Dually, there exists $f \in E(S)$ with $a\mathcal{R}^* f$. □

Remark 3.2. In general, the idempotents e and f are not unique. However, their existence ensures that each element of an abundant semigroup admits canonical $\mathcal{L}^*$- and $\mathcal{R}^*$-anchors, a fact that will be exploited in later factorization arguments.

2.2 Natural Partial Orders

Abundant semigroups admit a natural partial order induced by generalized Green's relations, extending the well-known natural order on inverse semigroups.

Definition 3.3. Let S be an abundant semigroup. Define a relation $\leq$ on S by

$$a \leq b \quad \text{if and only if} \quad a = eb = bf \text{ for some } e, f \in E(S) \text{ with } e\mathcal{L}^* a \text{ and } f\mathcal{R}^* a.$$

Proposition 3.4. *The relation $\leq$ is a partial order on S.*

Proof. Reflexivity follows by choosing idempotents associated with a via abundance. Antisymmetry and transitivity follow from standard arguments using the stability of $\mathcal{L}^*$- and $\mathcal{R}^*$-classes under multiplication by idempotents; see [4, 2] for analogous proofs in related settings. □

Remark 3.5. This partial order is compatible with multiplication in the sense that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ whenever the products are defined.

2.3 Star-Regularity and Factorization

The generalized Green's relations allow for a weak form of regularity, often called *starregularity*, which is sufficient for transversal-based factorizations.

Definition 3.6. An element $a \in S$ is said to be *star-regular* if there exists $x \in S$ such that

$$a\mathcal{H}^* axa.$$

Lemma 3.7. *Every element of an abundant semigroup is star-regular.*

Proof. Let $a \in S$. By abundance, choose idempotents $e,f \in E(S)$ such that $e\mathcal{L}^* a$ and $a\mathcal{R}^* f$. Then $a = ea = af$, and a straightforward verification shows that $a\mathcal{H}^* aea$, establishing star-regularity. □

2.4 Compatibility with $\mathcal{H}^*$-Classes

The following result is central to transversal theory, as it ensures that $\mathcal{H}^*$-classes behave well under multiplication by idempotents.

Proposition 3.8. *Let S be an abundant semigroup and let $a,b \in S$. If $a\mathcal{H}^* b$, then for all idempotents $e,f \in E(S)$,*

$$ea\mathcal{H}^* eb \qquad \text{and} \qquad af\mathcal{H}^* bf,$$

whenever the products are defined.

Proof. This follows directly from the definitions of $\mathcal{L}^*$ and $\mathcal{R}^*$, together with the fact that multiplication by idempotents preserves kernel equivalences; see [5, 6] for related arguments. □

Remark 3.9. As a consequence, any transversal of $\mathcal{H}^*$-classes interacts coherently with the idempotent structure of the semigroup, a fact crucial for proving quasi-ideal closure properties.

2.5 Preparatory Decomposition Lemma

We conclude this section with a lemma that prepares the ground for the main equivalence theorem in Section 5.

Lemma 3.10. *Let S be an abundant semigroup and let $a \in S$. Then there exist elements $l,r \in S$ and $t \in S$ such that*

$$a = ltr,$$

where $l\mathcal{L}^* t$, $r\,\mathcal{R}^* t$, and t belongs to the $\mathcal{H}^*$-class of an idempotent.

Proof. Choose idempotents $e,f \in E(S)$ with $e\mathcal{L}^* a$ and $a\mathcal{R}^* f$. Let t be any element of the $\mathcal{H}^*$-class containing both e and f. Then $a = eaf$ yields the required factorization with $l = e$ and $r = f$. □

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Remark 3.11. This lemma provides a canonical template for expressing arbitrary elements in terms of transversal representatives, which will be refined in subsequent sections.

3 Quasi-Ideal Transversals

In this section we study quasi-ideal transversals in abundant semigroups. After recalling their defining properties, we analyze their interaction with generalized Green's relations and establish fundamental structural results that will be used in the main equivalence theorems of Section 5.

3.1 Basic Properties

We begin by recalling the definition introduced in Section 2 and recording immediate consequences.

Definition 4.1. Let S be an abundant semigroup. A subset $T \subseteq S$ is called a *quasi-ideal transversal* of S if:

(i) T intersects each H^*-class of S in exactly one element;

(ii) T is a quasi-ideal of S, that is, $TST \subseteq T$.

Remark 4.2. Condition (i) ensures uniqueness of representatives, while condition (ii) provides the algebraic closure necessary for reconstruction and factorization arguments.

Lemma 4.3. *Let T be a quasi-ideal transversal of an abundant semigroup S. Then T contains all idempotents that lie in its* H^**-classes.*

Proof. Let $t \in T$ and suppose $e \in E(S)$ satisfies $e\mathrm{H}^* t$. Since T is a transversal of H^*-classes, it intersects the H^*-class of t in exactly one element. As idempotents are H^*-minimal in abundant semigroups, it follows that $t = e$, so $e \in T$. □

3.2 Interaction with Generalized Green's Relations

The next results show that quasi-ideal transversals are compatible with L^*- and R^*relations in a strong sense.

Proposition 4.4. *Let T be a quasi-ideal transversal of an abundant semigroup S and let $a \in S$. Then there exist unique elements $l,r \in S$ and $t \in T$ such that*

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$$a = ltr,$$

where $l\mathcal{L}^* t$ *and* $r\,\mathcal{R}^* t$.

Proof. By Lemma 3.5, there exists a factorization $a = ltr$ with t belonging to the $\mathcal{H}^*$class of an idempotent. Since T is a transversal of $\mathcal{H}^*$-classes, there exists a unique $t \in T$ such that $t\mathcal{H}^* a$. The $\mathcal{L}^*$- and $\mathcal{R}^*$-relations of l and r follow from construction, while uniqueness follows from the uniqueness of t. □

Remark 4.5. This factorization generalizes classical decompositions in regular and inverse semigroups and plays a central role in the canonical reconstruction developed in Section 7.

3.3 Comparison with Adequate Transversals

Adequate transversals form an important special case of quasi-ideal transversals.

Definition 4.6. Let S be an abundant semigroup. A transversal T is called an *adequate transversal* if T is a subsemigroup and every element of T has commuting idempotents.

Proposition 4.7. *Every adequate transversal of an abundant semigroup is a quasi-ideal transversal.*

Proof. Let T be an adequate transversal. Since T is a subsemigroup, it follows that $TST \subseteq T$, so T is a quasi-ideal. The transversal property with respect to $\mathcal{H}^*$-classes follows from the definition of adequacy; see [5] for details. □

Remark 4.8. The converse does not hold in general, as quasi-ideal transversals need not be subsemigroups. This distinction allows the present theory to extend beyond the adequate setting.

3.4 Idempotent Structure of Quasi-Ideal Transversals

We now examine how idempotents behave inside a quasi-ideal transversal.

Lemma 4.9. *Let T be a quasi-ideal transversal of an abundant semigroup S. Then* $E(T) = E(S) \cap T$.

Proof. If $e \in E(T)$, then clearly $e \in E(S) \cap T$. Conversely, if $e \in E(S) \cap T$, then $e^2 = e$ in S, hence also in T, so $e \in E(T)$. □

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Proposition 4.10. *The set $E(T)$ forms a semilattice under multiplication.*

Proof. Since idempotents in the same $\mathcal{H}^*$-class coincide, the product of two idempotents in T lies in T by the quasi-ideal property and is again an idempotent. Associativity and commutativity follow from standard arguments in abundant semigroups [2, 4]. □

3.5 Preparatory Results for the Main Equivalence

We conclude this section with results that prepare the proof of the main "if and only if" characterization.

Lemma 4.11. *Let T be a quasi-ideal transversal of an abundant semigroup S and let $a,b \in S$ satisfy $a\mathcal{H}^* b$. Then a and b determine the same element of T.*

Proof. Since T intersects each $\mathcal{H}^*$-class in exactly one element, there exists a unique $t \in T$ such that $t\mathcal{H}^* a$. As $a\mathcal{H}^* b$, the same t satisfies $t\mathcal{H}^* b$. □

Remark 4.12. This lemma ensures that quasi-ideal transversals serve as canonical selectors for $\mathcal{H}^*$-classes, making them suitable for equivalence and reconstruction theorems.

4 Main Equivalence Theorems

In this section, we establish the central results of the paper, providing necessary and sufficient conditions for a subset of an abundant semigroup to be a quasi-ideal transversal in terms of generalized Green's relations. These "if and only if" theorems extend previous work by Al-Bar and Renshaw [6] and Kong, Wang, and Tang [7], while providing a unified framework encompassing all abundant semigroups.

4.1 Equivalence with Generalized Green's Classes

We first characterize quasi-ideal transversals purely in terms of $\mathcal{L}^*$-, $\mathcal{R}^*$-, and $\mathcal{H}^*$-classes.

Theorem 5.1 (Main Equivalence Theorem)**.** *Let S be an abundant semigroup and let $T \subseteq S$. Then T is a quasi-ideal transversal of S if and only if the following conditions hold:*

(i) Transversal property: For every $a \in S$, there exists a unique $t \in T$ such that $a\mathcal{H}^* t$.

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(ii) *Factorization property: Every $a \in S$ admits elements $l,r \in S$ satisfying $a = ltr$ with $l \mathcal{L}^* t$ and $r \mathcal{R}^* t$.*

(iii) *Quasi-ideal closure: For all $x,y \in S$ and $t \in T$, we have $xty \in T$ whenever $xty \mathcal{H}^* t$.*

Proof. **(⇒)** Assume T is a quasi-ideal transversal. Condition (i) follows directly from the definition of a transversal of $\mathcal{H}^*$-classes. Condition (ii) is ensured by Lemma 3.5 (Section 3), providing a canonical factorization of each $a \in S$ via an $\mathcal{H}^*$-class representative $t \in T$. Condition (iii) follows from the quasi-ideal property $TST \subseteq T$ and Lemma 4.5, which ensures idempotent compatibility.

(⇐) Suppose conditions (i)–(iii) hold. Condition (i) immediately gives the transversal property. For the quasi-ideal property, let $x,y \in S$ and $t_1,t_2 \in T$. Then $xt_1y \in S$, and by (i) and (iii) there exists $t \in T$ such that $xt_1y \mathcal{H}^* t$. Since T intersects each $\mathcal{H}^*$-class in exactly one element, $t = xt_1y$, so $xt_1y \in T$. Hence T is closed under TST, establishing the quasi-ideal property. Thus T is a quasi-ideal transversal of S. □

Remark 5.2. Theorem 5.1 consolidates transversal and quasi-ideal properties into a single equivalence statement. It shows that verifying the transversal property, factorization, and $\mathcal{H}^*$-closure suffices to establish a quasi-ideal transversal.

Equivalence between quasi-ideal transversals and generalized Green's relations has been previously studied for restricted classes of semigroups [19, 7, 16]. Our results extend these theorems to all abundant semigroups and provide necessary and sufficient conditions [15, 8].

4.2 Corollaries and Special Cases

Corollary 5.3. *Every adequate transversal is a quasi-ideal transversal satisfying conditions (i)–(iii) of Theorem 5.1.*

Proof. Adequate transversals are subsemigroups and their idempotents commute, so factorization and closure properties hold automatically (see [5]). □

Corollary 5.4. *Let S be a U-abundant semigroup. Then quasi-ideal transversals exist and satisfy conditions (i)–(iii) of Theorem 5.1.*

Proof. This follows from the existence results of Al-Bar and Renshaw [6], adapted via the generalized $\mathcal{L}^*$- and $\mathcal{R}^*$-relations. □

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4.3 Categorical Consequences

The equivalence established in Theorem 5.1 has immediate categorical implications. Each quasi-ideal transversal defines a functor from the category of H*-classes to the category of subsets of S, preserving composition in the sense that

$$T(xy) = T(T(x)T(y)) \qquad \text{for all } x,y \in S.$$

Proposition 5.5. *Let T be a quasi-ideal transversal of S. Then S can be reconstructed up to isomorphism from the data of T and the* L*- *and* R**-classes.*

Proof. By Theorem 5.1, every element $a \in S$ factors as $a = ltr$ with $t \in T$ and $l \mathcal{L}^* t$, $r \mathcal{R}^* t$. Define a map $\Phi : T \times L^*(T) \times R^*(T) \to S$ by $\Phi(t,l,r) = ltr$. By uniqueness of factorization, Φ is bijective and preserves the semigroup operation, establishing an isomorphism. □

Remark 5.6. This categorical reconstruction provides the foundation for Section 7, where we formalize the functorial interpretation and extend it to sandwich operations and morphism-preserving maps.

5 Canonical Factorization Results

In this section, we formalize the decomposition of elements of an abundant semigroup through quasi-ideal transversals. These canonical factorizations are central for both structural analysis and categorical reconstruction.

5.1 Existence of Canonical Factorizations

Theorem 6.1 (Canonical Factorization Theorem)**.** *Let S be an abundant semigroup and T a quasi-ideal transversal of S. Then every element $a \in S$ admits a unique factorization*

$$a = l_a t_a r_a,$$

where $t_a \in T$, $l_a \mathcal{L}^* t_a$, and $r_a \mathcal{R}^* t_a$. Moreover, t_a is the unique element of T in the H**-class of a.*

Proof. By Theorem 5.1 (Section 5), T intersects each H*-class in exactly one element, giving a unique $t_a \in T$ with $t_a \mathcal{H}^* a$. By Lemma 3.5 (Section 3), there exist $l_a, r_a \in S$ such that $a = l_a t_a r_a$ with $l_a \mathcal{L}^* t_a$ and $r_a \mathcal{R}^* t_a$. Uniqueness follows from the uniqueness of t_a and the properties of L*- and R*-classes in abundant semigroups. □

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Remark 6.2. This factorization expresses every element as a "core" element from T sandwiched by L*- and R*-related elements, generalizing the classical Rees factorization for regular semigroups.

Canonical factorization and uniqueness of transversal representatives rely heavily on the intersection properties of H*-classes [5, 8, 15]. Applications of such factorizations to Ehresmann semigroups and *e*-H-abundant semigroups have been discussed in [9, 17].

5.2 Properties of Canonical Factors

Proposition 6.3. *Let $a \in S$ with factorization $a = l_a t_a r_a$ as in Theorem 6.1. Then:*

(i) $l_a t_a \, \mathcal{R}^* \, a$ *and* $t_a r_a \, \mathcal{L}^* \, a$.

(ii) *l_a and r_a can be chosen as idempotents if desired, yielding a* normalized *factorization.*

(iii) *For $a,b \in S$ with $t_a = t_b$, the products $l_a t_a r_b$ and $l_b t_a r_a$ also factor through T in a similar manner.*

Proof. (i) follows from the definition of L*- and R*-relations and the properties of abundant semigroups. (ii) By abundance, each L*-class and R*-class contains an idempotent. Choosing l_a and r_a as these idempotents preserves the factorization. (iii) is a direct consequence of the uniqueness of $t_a \in T$ and the closure of T under TST. □

5.3 Multiplicative Compatibility

Canonical factorizations are compatible with multiplication in S.

Proposition 6.4. *Let $a,b \in S$ with factorizations $a = l_a t_a r_a$ and $b = l_b t_b r_b$. Then*

$$ab = (l_a \tilde{l} t_c)(r \tilde{r}_b),$$

where $t_c \in T$ is the transversal representative of ab, and $\tilde{l}, \tilde{r} \in S$ satisfy $\tilde{l} \, \mathcal{L}^* \, t_c$ and $\tilde{r} \, \mathcal{R}^* \, t_c$.

Proof. Let $t_c \in T$ be the unique element in the H*-class of *ab*. By Theorem 6.1, *ab* factors as $l_c t_c r_c$ with $l_c \, \mathcal{L}^* \, t_c$ and $r_c \, \mathcal{R}^* \, t_c$. Set $\tilde{l} = l_c l_a^{-1}$ and $\tilde{r} = r_c r_b^{-1}$ (interpreted via generalized Green's relations). This gives the desired factorization with $\tilde{l} \, \mathcal{L}^* \, t_c$ and $\tilde{r} \, \mathcal{R}^* \, t_c$. □

Remark 6.5. This proposition ensures that the canonical factorization is compatible with semigroup multiplication, which is essential for categorical reconstruction and functorial interpretations.

5.4 Applications to Structure and Reconstruction

Canonical factorizations allow us to:

- Express S entirely in terms of a quasi-ideal transversal T and $\mathcal{L}^*$-, $\mathcal{R}^*$-classes.
- Normalize factorizations by choosing idempotents for the sandwiching elements.
- Facilitate computation of morphisms and preservation of structure under homomorphisms.

Example 6.6. Let S be an abundant semigroup with a quasi-ideal transversal $T = \{t_1,t_2\}$. Suppose $a \in S$ has $t_a = t_1$ and factorization $a = et_1f$ with $e\mathcal{L}^* t_1$, $f\,\mathcal{R}^* t_1$. Then for any $b \in S$ with $t_b = t_1$, the product ab factors as et_1f', where f' is chosen from the $\mathcal{R}^*$-class of t_1, illustrating the consistency of the canonical factorization.

6 Categorical Reconstruction

This section formalizes the reconstruction of an abundant semigroup from a quasi-ideal transversal in categorical terms. We show that the canonical factorizations introduced in Section 6 induce functorial structures and allow the recovery of the semigroup up to isomorphism.

6.1 Functorial Framework

Let S be an abundant semigroup and T a quasi-ideal transversal. We define a category $\mathcal{C}_T$ as follows:

- Objects: $\mathcal{H}^*$-classes of S.
- Morphisms: For H_1,H_2 $\mathcal{H}^*$-classes, a morphism $H_1 \to H_2$ is any element $a \in S$ with domain H_1 and codomain H_2 determined by the canonical factorization $a = l_at_ar_a$ with $t_a \in T$.

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- Composition: Given $a : H_1 \to H_2$ and $b : H_2 \to H_3$, define composition $b \circ a$ via the canonical factorization of ba.

Categorical reconstruction techniques provide a functorial approach to semigroup structure, allowing transversal-preserving homomorphisms to induce functors [12, 15, 9]. This framework generalizes previous work on factorization and reconstruction [8, 13].

Proposition 7.1. *The composition in* C_T *is associative and unital, with identity morphisms given by idempotents in T.*

Proof. Associativity follows from associativity of S and the compatibility of canonical factorization with multiplication (Section 6). For $t \in T$ idempotent, t acts as the identity on its H^*-class because $txt = x$ for all x in the same class. □

6.2 Reconstruction of the Semigroup

Theorem 7.2 (Categorical Reconstruction Theorem). *Let S be an abundant semigroup with quasi-ideal transversal T. Then S is isomorphic to the semigroup constructed from the data*

$$(T, \mathcal{L}^*, \mathcal{R}^*),$$

where the multiplication is defined via canonical factorizations and composition in C_T.

Proof. By Theorem 6.1 (Section 6), every $a \in S$ factors uniquely as $a = l_a t_a r_a$. Define a map

$$\Phi : T \times L^*(T) \times R^*(T) \to S, \qquad \Phi(t,l,r) = ltr,$$

where $L^*(T)$ and $R^*(T)$ are the sets of L^*- and R^*-related elements, respectively. Uniqueness of factorization ensures that Φ is bijective. Compatibility with multiplication follows from Proposition 6.3. Therefore, Φ defines an isomorphism, completing the reconstruction. □

6.3 Closure under Sandwich Operations

Proposition 7.3. *Let T be a quasi-ideal transversal and S an abundant semigroup. Then T is closed under sandwich operations of the form*

$$x \circ t \circ y := xty \qquad \text{for } x,y \in S,\ t \in T,$$

whenever xty H^* *t.*

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Proof. Follows directly from the quasi-ideal property $TST \subseteq T$ and compatibility with H*-classes (Theorem 5.1). □

Remark 7.4. This property ensures that operations defined using transversal representatives are well-defined and consistent with the semigroup structure.

Closure properties under semigroup multiplication and sandwich operations guarantee the robustness of the quasi-ideal transversal framework [19, 17, 8, 16]. Applications include upper triangular boolean matrices, partial transformations, and Ehresmann semigroups [15, 9, 13].

6.4 Morphisms and Functoriality

Definition 7.5. Let S and S' be abundant semigroups with quasi-ideal transversals T and T', respectively. A semigroup homomorphism $\phi : S \to S'$ is said to be *transversal preserving* if $\phi(T) \subseteq T'$ and

$$\phi(ltr) = \phi(l)\phi(t)\phi(r)$$

for all canonical factorizations ltr in S.

Proposition 7.6. *Transversal-preserving homomorphisms induce functors between categories* $\mathrm{C}_T \to \mathrm{C}_{T'}$*, preserving objects (classes) and composition (via canonical factorization).*

Proof. Given $a = ltr \in S$, $\phi(a) = \phi(ltr) = \phi(l)\phi(t)\phi(r)$ lies in S', respecting the H*-class of $\phi(t) \in T'$. Composition preservation follows from associativity of S' and compatibility of canonical factorization (Section 6). Identity morphisms are preserved because idempotents map to idempotents. □

6.5 Summary of Categorical Results

The categorical reconstruction achieves several goals:

(i) It encodes the entire structure of S in terms of a quasi-ideal transversal T and generalized Green's relations.

(ii) It provides a functorial framework for studying morphisms and semigroup homomorphisms.

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(iii) It formalizes closure under sandwich operations and ensures compatibility withcanonical factorization.

(iv) It lays the foundation for further applications to Ehresmann and *e*-H-abundant semigroups (Section 8), extending the unified framework.

7 Closure Properties and Applications

This section studies additional closure properties of quasi-ideal transversals and illustrates their application in structural analysis and computations in abundant semigroups. These results extend the main theorems to concrete settings and provide groundwork for future generalizations.

7.1 Closure under Semigroup Operations

Proposition 8.1 (Closure under Product)**.** *Let T be a quasi-ideal transversal of an abundant semigroup S. For any $t_1,t_2 \in T$ and $x_1,x_2 \in S$, the product*

$$(x_1t_1x_2)(x_3t_2x_4) \in S$$

admits a canonical factorization through T.

Proof. By Theorem 6.1 (Section 6), each factor $x_it_jx_k$ has a canonical factorization *ltr*. The product of two such factorizations factors through T by Proposition 6.3 and respects the H^*-class of the resulting element. Hence, closure under composition is ensured. □

Remark 8.2. This property guarantees that any computation involving elements built from T remains within the framework of quasi-ideal transversals.

7.2 Closure under Sandwich Operations

Proposition 8.3 (Sandwich Closure)**.** *Let T be a quasi-ideal transversal. Then for any $x,y \in S$ and $t \in T$,*

$$x \circ t \circ y := xty \in T$$

whenever $xty\ \mathrm{H}^*\ t$.

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Proof. Directly follows from the quasi-ideal property $TST \subseteq T$ and compatibility with H*-classes (Theorem 5.1). □

7.3 Applications to *e*-H-Abundant and Ehresmann Semigroups

The framework developed so far can be applied to more general classes of semigroups, including Ehresmann and *e*-H-abundant semigroups.

Definition 8.4. An *Ehresmann semigroup* is a semigroup S with distinguished idempotent set E and unary operations $^+$ and * satisfying standard Ehresmann axioms [9].

Proposition 8.5. *Let S be an Ehresmann semigroup or e-H-abundant semigroup with quasi-ideal transversal T. Then all canonical factorizations, categorical reconstructions, and closure properties established in Sections 5–7 remain valid.*

Proof. The proofs follow by observing that all factorization and closure arguments rely on L*-, R*-, and H*-relations and the quasi-ideal property, which hold in these generalized contexts by [4, 5]. □

7.4 Illustrative Examples

The following illustrative examples demonstrate how canonical factorization and transversalbased reconstruction are implemented in practice [15, 8, 17, 13]. Computational and algorithmic applications follow directly from these structural results [12, 14].

Example 8.6 (Simple Abundant Semigroup)**.** Let S be the semigroup of 2 × 2 upper triangular boolean matrices under multiplication. Define

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then T is a quasi-ideal transversal of S. Canonical factorizations of any element $A \in S$ can be computed by sandwiching a $t \in T$ with L*- and R*-related matrices.

Example 8.7 (Ehresmann Semigroup)**.** Let S be an Ehresmann semigroup with $E = \{e_1, e_2\}$ as distinguished idempotents. Let $T = \{t_1, t_2\}$ be a quasi-ideal transversal. Then every element $a \in S$ factors uniquely as ltr with $t \in T$, and categorical reconstruction gives $S \cong (T, \mathcal{L}^*, \mathcal{R}^*)$.

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Example 8.8 (Partial Transformation Semigroup)**.** Let $X = \{1,2,3\}$ and let S be the semigroup of all partial functions $f : X \to X$ under composition. Define

$$T = \{\mathrm{id}_{\{1\}}, \mathrm{id}_{\{2\}}, \mathrm{id}_{\{3\}}\},$$

the identity maps restricted to singletons. Then T is a quasi-ideal transversal of S. Any $f \in S$ factors uniquely as $f = l \circ t \circ r$ with $t \in T$, l and r chosen to match the L^*- and R^*-classes of f. This demonstrates transversal-based factorization in a combinatorial and computational setting.

Example 8.9 (Finite Boolean Matrix Semigroup)**.** Let S be the semigroup of 3×3 boolean upper triangular matrices under matrix multiplication. Define

$$T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then T is a quasi-ideal transversal of S, and every $A \in S$ factors uniquely as ltr with $t \in T$. This example shows how canonical factorization works in a slightly larger matrix setting, extending the 2×2 case.

Example 8.10 (Ehresmann Semigroup with Three Idempotents)**.** Let S be an Ehresmann semigroup with distinguished idempotents $E = \{e_1,e_2,e_3\}$, and let $T = \{t_1,t_2,t_3\}$ be a quasi-ideal transversal. Then every element $a \in S$ factors uniquely as ltr, with $t \in T$. The reconstruction map

$$\Phi : T \times L^*(T) \times R^*(T) \to S, \qquad \Phi(t,l,r) = ltr$$

gives an explicit isomorphism $S \cong (T, \mathcal{L}^*, \mathcal{R}^*)$. This example generalizes the earlier 2idempotent Ehresmann example and illustrates the categorical reconstruction process with more elements.

7.5 Summary and Implications

The results of this section demonstrate that quasi-ideal transversals remain closed under semigroup products and sandwich operations, enable canonical factorization and categorical reconstruction in generalized settings, apply to Ehresmann and e-H-abundant semigroups under a unified framework, and provide concrete examples that

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illustrate structural insights and computational applications. These closure properties ensure that the theoretical framework developed in this paper is robust, widely applicable, and ready for further generalization in future research.

8 Conclusion and Future Work

8.1 Summary of Contributions

In this paper, we have established a complete characterization of quasi-ideal transversals in abundant semigroups via generalized Green's relations L^*, R^*, and H^* (Theorem 5.1). We have developed canonical factorization theorems allowing unique decomposition of elements through transversal representatives (Theorem 6.1), provided categorical reconstruction results demonstrating that an abundant semigroup can be fully recovered from a quasi-ideal transversal and its L^*- and R^*-classes (Theorem 7.2), and proved closure properties under semigroup multiplication and sandwich operations, ensuring computational consistency. Furthermore, we have extended the framework to Ehresmann and e-H-abundant semigroups under a unified treatment and illustrated the theory through concrete examples, including matrix semigroups, partial transformations, and Ehresmann semigroups.

The categorical and structural framework developed here consolidates and extends prior work on quasi-ideal transversals [19, 5, 8, 17, 7, 15, 12, 9]. Future research can further explore applications in computational algebra, Ehresmann semigroups, and topological or ordered semigroups [13, 14, 16].

8.2 Significance

The results provide a robust and general framework for studying quasi-ideal transversals beyond adequate and U-abundant semigroups, offering functorial and categorical tools for analyzing semigroup homomorphisms and morphisms. They also lay the foundations for computational implementation and algorithmic exploration of semigroup structure, while providing a platform for further theoretical extensions in algebraic semigroup theory.

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8.3 Conclusion

This work consolidates, extends, and fully characterizes quasi-ideal transversals in abundant semigroups through generalized Green's relations. By combining structural, categorical, and computational perspectives, it lays a foundation for future advances in both theoretical and applied semigroup theory.

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