BOUNDEDNESS OF GENERALLY FRACTIONAL INTEGRAL OPERATOR ON GENERAL MORREY SPACE

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ABSTRACT. In this study I will discuss the limits of fractional integral operators in the homogeneous and nonhomogeneous Lebesgue space, the Morrey space and the general Morrey space. In particular, in this study it will be proven that the fractional integral boundaries formulated in the Morrey space are generally not homogeneous. Evidence of integral fractional boundaries formulated in the Morrey space is generally not homogeneous using the specified maximum operator properties in space and using Hedberg's inequality. This evidence is an extension of Hardy-Littlewood-Sobolev's inequality [11, 22]. My research related to BOUNDEDNESS OF GENERALLY FRACTIONAL INTEGRAL OPERATOR ON GENERAL MORREY SPACE as a scientific work that must be published in an international journal, as for the results I present in this journal, is the result of research

I. Introduction

Suppose that α ∈R and 0 <α <n. The fractional integral operator or potential Riesz Iα is

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|}$$

−α *dy* (1) *n*

for every x ∈ Rn. Size μ which satisfies the condition of growth, ie there are c> 0 and 0 <n ≤ d so that μ (B (x, r)) ≤ crn (2)for each ball centered on x ∈Rd and has radius r> 0 then (Rd, μ) is called a non-homogeneous space. In nonhomogeneous space, fractional integral operators are defined

as with $I^n_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y)$

for 0 <α <n ≤ d and x ∈ Rd. It can be seen that if n = d and μ are Lebesgue sizes then they are obtained.

In [5], it is proven that, if 1 and, for 0 <α <n

Then it is limited from Lebesgue non-homogeneous space Lp (μ) to Lq (μ). Furthermore, in the wider space of the limited Lebesgue space from the non-homogeneous Morrey space which is generally Lp, φ (μ) to Lq, ψ (μ). For any function f measured-μ with μ borel size on Rd that satisfies the condition of growth (2), for 1 ≤ p <∞ and φ: (0, ∞) → (0, ∞) Morrey space is generally Lp, φ (μ) = Lp, φ (Rd, μ) is Lp, φ (μ) = {f ∈ Lploc (μ): ‖ f ‖ Lp, φ (μ) <∞} with

$$||f||_{L^{p,\phi}(\mu)} := \sup_{B(a,r)} \left[\frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(a,r)} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \right] < \infty.$$

for 0 <α <n ≤ d and x ∈ Rd. It can be seen that if n = d and μ are Lebesgue sizes then they are obtained.In [5], it is proven that, if 1 and, for 0 <α <nthen it is limited from Lebesgue non-homogeneous space Lp (μ) to Lq (μ). Furthermore, in the wider space of the limited Lebesgue space from the non-homogeneous Morrey space which is generally Lp, φ (μ) to Lq, ψ (μ). For any function f measured-μ with μ borel size on Rd that satisfies the condition of growth (2), for 1 ≤ p <∞ and φ: (0, ∞) → (0, ∞) Morrey space is generally Lp, φ (μ) = Lp, φ (Rd, μ) is Lp, φ (μ) = {f ∈ Lploc (μ): ‖ f ‖ Lp, φ (μ) <∞} with

With the function φ is a positive function where φ: (0, ∞) → (0, ∞) which must fulfill the following two conditions,

1. The function φ (r) is almost down, namely there is a constant C> 0 such that for each r r s applies φ (r) ≥ Cφ (s).

2. The function rαφ (r) p almost rises, that is, there is a constant C> 0 such that for each r≤s applies rαφ (r) p ≤ Csαφ (s) p.

Because both of these requirements must be fulfilled by the function φ this function fulfills the doubling condition, namely there is a constant C> 0 such that if 2 then, for each r, s> 0. Note Proposition 1 and Lemma 2 below. Proposition 1. Suppose that ω is a non-negative function and f is neutralized locally at Rd, for 1 <p <∞, then there is c> 0 so that

$$\int_{\mathbb{R}^d} |M^{\mu} f(x)|^p \omega(x) d\mu(x) \leq c \int_{\mathbb{R}^d} |f(x)|^p M^{\mu} \omega(x) d\mu(x). \qquad (3)$$

The above inequality is called the Fefferman-Stein inequality and the proof can be seen in [22] page 29.

Lemma 2. If the function φ: (0, ∞) → (0, ∞) satisfies the doubling condition then

$$\phi\left(2^{k+1}r\right)^{p} \sim \int_{2^{k}r}^{2^{k+1}r} \frac{\phi(t)^{p}}{t}dt$$

for every r> 0 and k positive integers.

Based on Proposition 1 and Lemma 2 it can be shown that the maximum operator Mμ is defined as

$$M^{\mu}f(x) := \sup_{r>0} \frac{1}{r^{n}} \int_{B(0,r)} |f(x-y)|d\mu(y)$$

for x ∈Rd and f ∈ L1loc (Rd), limited to Lp, φ (μ) for 1 <p <∞ (see [16], page 8) stated in the following theorem.

Theorem 3. Suppose f is integrally localized at Rd, φ: (0, ∞) → (0, ∞) satisfies doubling conditions and for a c1> 0

$$\int_{r}^{\infty} \frac{\phi(t)^{p}}{t}dt \leq c_{1}\,\phi(r)^{p}$$

for every r> 0 and 1 ≤ p <∞, then $\|M^{\mu}f\|_{L^{p,\varphi}(\mu)} \leq C\,\|f\|_{L^{p,\varphi}(\mu)}$ (4)

for a C> 0.

Evidence. Take any f ∈ Lp, φ (μ) and B (a, r) are open balls centered on a ∈Rd and radius r> 0 so ω = χB (a, r) is a non-negative function. Then according to equality (3) is obtained,

$$\int_{B(a,r)} |M^{\mu}f(x)|^{p}d\mu(x)$$

$$\leq \int_{\mathbb{R}^{d}} |M\mu f(x)|^{p}\chi_{B(a,r)}d\mu(x)$$

$$\leq C\int_{\mathbb{R}^{d}} |f(x)|^{p}M\mu\chi_{B(a,r)}d\mu(x)$$

$$\leq C\left[\int_{B(a,r)} |f(x)|^{p}d\mu(x) + \sum_{k=1}^{\infty}\int_{B(a,2^{k+1}r)-B(a,2^{k}r)} |f(x)|^{p}M^{\mu}\chi_{B(a,r)}d\mu(x)\right]. \quad (5)$$

Next, for $x \in B\left(a, 2^{k+1}r\right) - B\left(a, 2^k r\right)$, we are estimated $M^{\mu}\chi_{B(a,r)}(x)$ as follows.

$$\begin{aligned} M^{\mu}\chi_{B(a,r)}(x) &= \sup_{R>0} \frac{1}{R^n} \int_{B(o,r)} |\chi_{B(a,r)}(x-y)| d\mu(x) \\ &= \sup_{R>0} \frac{1}{R^n} \mu[B(a,r) \cap B(x,R)] \\ &= \frac{\mu[B(a,r)]}{(|x-a| + 2^k r)^n} \\ &\leq \frac{\mu[B(a,r)]}{(2^k r)^n} \\ &= \frac{cr^n}{2^{kn} r^n} \\ &= \frac{c}{2^{kn}}. \end{aligned}$$

Based on (5) obtained,

Z

$$\begin{aligned} &\leq C\left[\int_{B(a,r)} |f(x)|^p d\mu(x)\right] + \\ &+ C\left[\sum_{k=1}^{\infty} \frac{1}{2^{kn}} \int_{B\left(a,2^{k+1}r\right) - B\left(a,2^k r\right)} |f(x)|^p d\mu(x)\right] \\ &\leq C\left[|B(a,2r)|\phi(2r)^p + \sum_{k=0}^{\infty} |B(a,2^{k+1}r)|\phi(2^{k+1}r)^p\right] ||f||_{L^{p,\phi}(\mu)} \\ &\leq C\, r^n ||f||_{L^{p,\phi}(\mu)} \left[\phi(r)^p + \sum_{k=0}^{\infty} \phi(2^{k+1}r)^p\right] \\ &\leq C\, r^n ||f||_{L^{p,\phi}(\mu)} \left[\phi(r)^p + \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\phi(t)^p}{t} dt\right] \\ &\leq C\, r^n ||f||_{L^{p,\phi}(\mu)} \left[\phi(r)^p + \int_r^{\infty} \frac{\phi(t)^p}{t} dt\right] \\ &\leq C\, r^n ||f||_{L^{p,\phi}(\mu)} \phi(r)^p. \end{aligned}$$

$|M^{\mu}f(x)|^p d\mu(x) B(a,r)$

So, got it

$$\sup_{B(a,r)} \frac{1}{\phi(r)} \left[\frac{1}{r^n} \int_{B(a,r)} |M^{\mu} f(x)|^p d\mu(x)\right]^{\frac{1}{p}} \leq C||f||_{L^{p,\phi}(\mu)}$$

There for $\|M^{\mu}f\|_{Lp,\varphi(\mu)} \leq C \|f\|_{Lp,\varphi(\mu)}$ Maximum operator limitation Mμ above is needed in proving the boundedness of fractional integral operators and fractional integral operators commonly from the Morrey space are generally Lp, φ to the Morrey space is generally for 1 <p <q <∞ with $p\ \psi = \varphi^q$ [3, 5].

II. DISCUSSION

Fractional integral operators here are generally fraction (integral) integrals using the function ρ, which is a non-negative function, namely ρ: (0, ∞) → (0, ∞) (also φ and ψ) and satisfies doubling conditions. For 0 <n ≤ d and the function ρ: (0, ∞) → (0, ∞) fractional integrals are generally Iρμ in nonhomogeneous space defined as

$$I_\rho^\mu f(x) := \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y).$$

Lemma 4. Suppose φ: (0, ∞) → (0, ∞) with lim φ (R) = ∞ and lim φ (R) =
R → 0 + R → ∞

0 and fulfills doubling conditions so for every t ∈ R, t> 0 there is R> 0 so that

$$\phi(R) < t \leq \phi\left(\frac{R}{2}\right). \quad (6)$$

Theorem 5. Suppose φ doubling and fulfilling

1. $\int_r^\infty \frac{\phi(t)^p}{t} dt \leq c_1 \phi(r)^p,\ r > 0, c_1 > 0$

   and inequality

2. $\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq c_2\, \phi^{\frac{p}{q}}(r),\ r > 0,\ c_2 > 0$

Where 1 < p < q <∞, then

$$\|I_\rho^\mu f\|_{L^{q,\varphi}(\mu)} \leq C \|f\|_{L^{p,\varphi}(\mu)}.$$

Evidence. For each x ∈Rd and R> 0, we write

$$\begin{aligned} I_\rho^\mu f(x) &= \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y) + \int_{|x-y|\geq R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y) \\ &= I_1(x) + I_2(x). \end{aligned}$$

Note for I1 (x), obtained

$$
\begin{aligned}
|I_1(x)| &\leq \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} f(y)d\mu(y) \\
&\leq \sum_{k=-\infty}^{-1} \int_{2^kR\leq|x-y|\leq 2^{k+1}R} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|d\mu(y) \\
&\leq C\sum_{k=-\infty}^{-1} \frac{\rho(2^kR)}{(2^kR)^n}\int_{|x-y|<2^{k+1}R}|f(y)|d\mu(y) \\
&\leq CM^\mu f(x)\sum_{k=-\infty}^{-1}\rho(2^kR) \\
&\leq CM^\mu f(x)\sum_{k=-\infty}^{-1}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)}{t}dt \\
&\leq CM^\mu f(x)\int_0^R \frac{\rho(t)}{t}dt \\
&\leq CM^\mu f(x)\phi(R)^{\frac{(p-q)}{q}}.
\end{aligned}
$$

Next, for I2 (x) is obtained,

$$
\begin{aligned}
|I_2(x)| &\leq \int_{|x-y|\geq R} \frac{\rho(|x-y|)}{|x-y|^n} f(y)d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \int_{2^kR\leq|x-y|\leq 2^{k+1}R} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|d\mu(y) \\
&\leq C\sum_{k=0}^{\infty} \frac{\rho(2^kR)}{(2^kR)^n}\int_{|x-y|<2^{k+1}R}|f(y)|d\mu(y) \\
&\leq C\sum_{k=0}^{\infty} \frac{\phi(2^{k+1}R)\rho(2^kR)}{(2^kR)^{n-\frac{n}{p}}}\left(\frac{1}{\phi(2^{k+1}R)}\left[\frac{1}{(2^{k+1}R)^n}\int_{|x-y|<2^{k+1}R}|f(y)|d\mu(y)\right]^{\frac{1}{p}}\right) \\
&\leq C||f||_{L^{p,\phi}(\mu)}\sum_{k=0}^{\infty}\rho(2^{k+1}R)\phi(2^{k+1}R) \\
&\leq C||f||_{L^{p,\phi}(\mu)}\sum_{k=0}^{\infty}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)\phi(t)}{t}dt \\
&\leq C||f||_{L^{p,\phi}(\mu)}\int_R^{\infty}\frac{\rho(t)\phi(t)}{t}dt \\
&\leq C||f||_{L^{p,\phi}(\mu)}\phi(R)^{\frac{p}{q}}.
\end{aligned}
$$

By adding I1 and I2, obtained

$$
|I_\rho^\mu f(x)| \leq C\left[M^\mu f(x)\phi(R)^{\frac{(p-q)}{q}} + ||f||_{L^{p,\phi}(\mu)}\phi(R)^{\frac{p}{q}}\right]. \tag{7}
$$

Next, assuming f 6 = 0, suppose 0. Based on (4),

$$
\phi(R) < \frac{M^\mu f(x)}{||f||_{L^{p,\phi}(\mu)}} \leq \phi\left(\frac{R}{2}\right).
$$

As a result,

$$|I_\rho^\mu f(x)| \leq C \left[M^\mu f(x) \left(\frac{M^\mu f(x)}{||f||_{L^{p,\phi}(\mu)}} \right)^{\frac{p}{q}-1} + ||f||_{L^{p,\phi}(\mu)} \left(\frac{M^\mu f(x)}{||f||_{L^{p,\phi}(\mu)}} \right)^{\frac{p}{q}} \right]$$
$$\leq C \left[M^\mu f(x)^{\frac{p}{q}} ||f||_{L^{p,\phi}(\mu)}^{1-\frac{p}{q}} \right]$$

for every x. Thus obtained,

$$\int_{B(a,r)} |I_\rho^\mu f(x)|^q d\mu(x) \leq \int_{B(a,r)} ||f||_{L^{p,\phi}(\mu)}^{q-p} M^\mu f(x)^p d\mu(x)$$
$$\leq C ||f||_{L^{p,\phi}(\mu)}^{q-p} \int_{B(a,r)} M^\mu f(x)^p d\mu(x).$$

So,

$$\frac{1}{\phi(r)^{\frac{p}{q}}} \left(\frac{1}{r^n} \int_{B(a,r)} |I_\rho^\mu f(x)|^q d\mu(x) \right)^{\frac{1}{q}} \leq \frac{C}{\phi(r)^{\frac{p}{q}}} \left(\frac{1}{r^n} ||f||_{L^{p,\phi}(\mu)}^{q-p} \int_{B(a,r)} M^\mu f(x)^p d\mu(x) \right)^{\frac{1}{q}}$$
$$\leq C ||f||_{L^{p,\phi}(\mu)}^{1-\frac{p}{q}} \left(\frac{1}{r^n} \frac{1}{\phi(r)^p} \int_{B(a,r)} M^\mu f(x)^p d\mu(x) \right)^{\frac{1}{q}}$$
$$\leq C ||f||_{L^{p,\phi}(\mu)}^{1-\frac{p}{q}} \left(||M^\mu f(x)||_{L^{p,\phi}(\mu)} \right)^{\frac{p}{q}}$$
$$\leq C ||f||_{L^{p,\phi}(\mu)}^{1-\frac{p}{q}} \left(||f(x)||_{L^{p,\phi}(\mu)} \right)^{\frac{p}{q}}$$
$$\leq C ||f||_{L^{p,\phi}(\mu)}.$$

As a result,

$$||I_\rho^\mu f||_{L^{q,\phi^{\frac{p}{q}}}(\mu)} \leq C ||f||_{L^{p,\phi}(\mu)}.$$

Thus, it is evident that the generalized integral fractional operator is also bounded in the Morrey space which is generally not homogeneous.

III. CONCLUSION

It can be seen that if the function ρ (t) = tα is chosen then for each x, y ∈Rd applies ρ (| x - y |) = | x - y | α, consequently

$$
\begin{aligned}
I_\rho^\mu f(x) &:= \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y) \\
&= \int_{\mathbb{R}^d} \frac{|x-y|^\alpha}{|x-y|^n} f(y) d\mu(y) \\
&= \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y) \\
&= I_\alpha^n f(x).
\end{aligned}
$$

Thus, the boundedness of the fractional integral operators that are generally formulated in the Morrey space are not homogeneous resulting in the boundedness of fractional integral operators in the Morrey space which are generally not homogeneous. In addition, if dμ = dx then it results in the limitation of the fractional integral operator Iα in the Morrey space. Next, with the selection of functions, for each f ∈ Lp, φ (Rd) is obtained,

$$
\begin{aligned}
||f||_{L^{p,\phi}(\mu)} &:= \sup_{B(x,r)} \left[\frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \right] \\
&= \sup_{B(x,r)} \left[\frac{1}{r^{\frac{\lambda-n}{p}}} \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \right] \\
&= \sup_{B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\
&= ||f||_{L^{p,\lambda}(\mu)}.
\end{aligned}
$$

Thus, if then Lp, φ (μ) = Lp, λ (μ). Also, if selected φ (t) = then Lp, φ (μ) = Lp (μ), whereas if dμ = dx, for Lp, φ (μ) = Lp, λ (Rn) and for Lp, φ (μ) = Lp (Rn).

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