Effectiveness similar transposed set of polynomials of two complex variables in different regions

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Abstract

In this paper we derive the effectiveness of similar transposed sets of polynomials of two complex variables in origin, when the constituent sets are originally effective under a normalizing conditions for these sets.
Moreover, when the constituent sets under the normalizing conditions are algebraic and functional sets, the effectiveness of similar transposed sets of polynomials in open hyperspheres is given here. Finally the effectiveness of similar transposed sets of polynomials and effectiveness of inverse similar transposed sets of polynomials are studied here.

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1. Introduction and Preliminaries

In recent decades, we have centered our attention on a new family of multivariate polynomials, and similar sets of polynomials, which are a great example of using operational techniques in a general setting.
So we will first present similar sets of polynomials and then summarise some basic findings related to these two families of bivariate polynomials which provide the main background in our analysis along with the principle of analytical functions in [4, 10].
Polynomial sequences play an important role in solving numerous problems that exist in many different fields of pure and applied mathematics (see, for example, [3, 11, 22]).
In 1937 Cannon [2] introduced convergence of some polynomials among the many polynomials.
The main objective of this paper is to study the effectiveness similar transposed sets of polynomials of one complex variable, which was recently defined and studied by Sayyed and Mena [18, 19], Sayyed and Metwally [20, 21].
Newns [15] was introduced the transposed inverse set of a given basic set of polynomials is the set whose matrix of coefficients is the transposed inverse of that of the given set. Adepoju [1; Chapter II]), introduced the effectiveness properties, in Faber regions, of the transposed inverse set of a given basic set of polynomials.
In addition, similar sets of polynomials of two complex variables were defined and studied by Makky [8, 9], a sequence $\{p_{m,n}(z,w)\}$ being a basic set of polynomials of two complex variables z and w is said to form a basic set, if the monomial $z^m w^n;\ m,n \geq 0$ for a unique finite representation as follows (see [2,5, 13]):

$$z^m w^n = \sum_{(h,k)=0}^{(m,n)} \pi_{m,n;h,k}\, p_{h,k}(z,w) \tag{1.1}$$

and the polynomials $\{p_{m,n}(z,w)\}$ are expressed in polynomial form as follows:

$$p_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} p_{m,n;h,k}\, z^h w^k . \tag{1.2}$$

The values $p_{m,n}^{h,k}$ and $\bar{p}_{m,n}^{h,k}$ are called matrices of coefficients and operators of the basic set $\{p_{m,n}(z,w)\}$ respectively; each of which is row finite. Thus, the necessary and sufficient condition for the set $\{p_{m,n}(z,w)\}$ to be basic if

$$p_{m,n}^{m,n} = \bar{p}_{m,n}^{m,n} = I$$

where I is an infinite unit matrix and (m,n) $=\frac{1}{2}$(m+n)(m+n+1)+n.

Let $\{p_{m,n}^{(i)}(z,w)\}$; $i=1,2$ where $p_{m,n}^{(i)}(z,w) = \sum_{(h,k)=0}^{(m,n)} p_{m,n}^{h,k} z^h w^k$; $i=1,2$ two basic sets of polynomials of two complex variables be. Also, the matrices coefficients and operators $p^{(i)} = \left(p_{m,n}^{(i)h,k}\right)$, $\bar{p}^{(i)} = \left(\bar{p}_{m,n}^{(i)h,k}\right)$ are arranged according to the sequence of double suffices entities $(e_{i,j})$ as follows $e_{0,0}, e_{1,0}, e_{0,1}, e_{2,0}, e_{1,1}, e_{0,2}, ...$; the value (i,j) for the enumerator number of $\sigma_{i,j}$ among this sequence, such that:

$$(i,j) = \frac{1}{2}(i+j)(i+j+1)+j \quad ; \quad (i,j) > 0 .$$

The basic set $\{p_{m,n}(z,w)\}$ of polynomials will be called simple set if the polynomials $p_{m,n}(z,w)$ are of order n, if

$$p_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} p_{m,n}^{h,k} z^h w^k$$

and it is a monic set if $p_{m,n}^{m,n} = 1$ for all (m,n), a basic set $\{p_{m,n}(z,w)\}$ of polynomials is said to be Cannon set, if the number; $N_{m,n}$; of non -zero elements in the relation (1.1) holds

$$\lim_{m+n\to\infty} \{N_{m,n}\}^{\frac{1}{m+n}} = 1 ,$$

otherwise it is called a general basic set (see e.g. [2]).

Also, the basic set $\{p_{m,n}(z,w)\}$ is said to be algebraic of degree N; when its matrix of coefficients satisfies the usual identity in [13] as follows:

$$a_0 p^N + a_1 p^{N-1} + ... + a_N I = 0 .$$

The Cannon sum $\omega_{m,n}[r]$; of the general basic set $\{p_{m,n}(z,w)\}$ is given by (see [14, 16, 17])

(1.3) $$\omega_{m,n}[r] = \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} |\bar{p}_{m,n}^{h,k}| M\left[\bar{p}_{m,n}; r\right]$$

and the Cannon function for the same set is

(1.4) $$\omega[r] = \limsup_{m+n\to\infty} \{\omega_{m,n}(z,w)\}^{\frac{1}{m+n}} .$$

Also, suppose that $\{\bar{p}_{m,n}(z,w)\}$ be inverse set of polynomials of the set $\{p_{m,n}(z,w)\}$ where

(1.5) $$\bar{p}_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} \bar{p}_{m,n}^{h,k} z^h w^k$$

and

$$z^m w^n = \sum_{(h,k)=0}^{(m,n)} p_{h,k}^{m,n} \overline{p}_{h,k}(z,w).$$

Let $\{p_{m,n}^{(i)}(z,w)\}$; $i=1,2$ are two basic sets of polynomials and set $\{p_{m,n}(z,w)\}$ is called the product set of two sets $\{p_{m,n}^{(i)}(z,w)\}$; $i=1,2$ (see [1, 11, 12, 13]),

$$\{p_{m,n}(z,w)\} = \{p_{m,n}^{(1)}(z,w)\}\{p_{m,n}^{(2)}(z,w)\},$$

$$p_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} p_{h,k}^{m,n} z^h w^k = \sum_{(h,k)=0}^{(m,n)} \sum_{(s,t)=0}^{(h,k)} p_{m,n}^{(1)s,t} p_{s,t}^{(2)h,k} z^h w^k .$$

Makky in [9] study effectiveness the similar sets of polynomials of a single complex variable when each of the constituent sets is basic.

Now, consider similar sets of polynomials of two complex variables, whenever each of the constituent sets are transposed basic sets.

Definition:

Assume that $\{\tilde{p}_{m,n}^{(i)}(z,w)\}$; $i=1,2$ be a transposed basic sets of polynomials; and let $\{u_{m,n}(z,w)\}$ a basic set of polynomials given by (see [14, 18])

(1.7) $$\{u_{m,n}(z,w)\} = \{\tilde{p}_{m,n}^{(1)}(z,w)\}\{\tilde{q}_{m,n}^{(2)}(z,w)\}\{\widehat{p}_{m,n}^{(1)}(z,w)\}$$

where

(1.8) $$u_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} u_{m,n;h,k} z^h w^k$$

and

$$u_{m,n}^{h,k} = \sum_{(s,t)=0}^{(m,n)} \sum_{(i,j)=0}^{(s,t)} \sum_{(h,k)=0}^{(i,j)} \tilde{p}_{m,n}^{(1)s,t} \tilde{p}_{s,t}^{(2)i,j} \widehat{p}_{i,j}^{(1)h,k} z^h w^k .$$

That can also, be written in the form

$$u_{m,n}^{h,k} = \sum_{(s,t)=0}^{(m,n)} \sum_{(i,j)=0}^{(s,t)} \sum_{(h,k)=0}^{(i,j)} p_{s,t}^{(1)m,n} p_{i,j}^{(2)s,t} \overline{p}_{h,k}^{(1)i,j} z^h w^k .$$

Then the set $\{u_{m,n}(z,w)\}$ is called a similar transposed set of polynomials of two complex variables (see .e.g. [7]).

Configuring the basic property for similar transposed sets while they $\{\tilde{p}_{m,n}^{(i)}(z,w)\}$; $i=1,2$ are basic. Also, let $\tilde{p}^{(i)}$; $i=1,2$, are matrices of coefficients of the sets $\{\tilde{p}_{m,n}^{(i)}(z,w)\}$; $i=1,2$, $\tilde{p}^{(i)} = \left(\tilde{p}_{m,n}^{(i)h,k}\right)$ and the values U, $\overline{U}$ are matrices coefficients of the similar transposed sets $\{u_{m,n}(z,w)\}$.

Write the matrices $U = \tilde{p}^{(1)}\tilde{p}^{(2)}\widehat{p}^{(1)}$ and $\overline{U} = \tilde{p}^{(1)}\tilde{p}^{(2)}\widehat{p}^{(1)}$, then we get

$$U\overline{U} = \tilde{p}^{(1)}\tilde{p}^{(2)}\widehat{p}^{(1)}\tilde{p}^{(1)}\tilde{p}^{(2)}\widehat{p}^{(1)} = I$$

and

$$\overline{U}\,U = \tilde{p}^{(1)}\widehat{p}^{(2)}\widehat{p}^{(1)}\tilde{p}^{(1)}\tilde{p}^{(2)}\widehat{p}^{(1)} = I$$

where I is unit infinite matrix.

Hence the matrix U of coefficients of the set $\{u_{m,n}(z,w)\}$ has a unique inverse $\overline{U}$, therefore the set $\{u_{m,n}(z,w)\}$ is basic.

2- Effectiveness of similar transposed set of polynomials at the origin

In this section we study the effectiveness of a similar transposed set of polynomials $\{u_{m,n}(z,w)\}$ two complex variables, at the origin with normalizing conditions and $\{p_{m,n}^{(i)}(z,w)\}$; $i=1,2$ fulfil the following conditions:

(2.1) $$\mu^{(i)}[0+]=0$$

(2.2) $$\nu^{(i)}[0+]>0; \qquad \forall\, r \succ 0$$

where

(2.3) $$\mu^{(i)}[0+]=\limsup_{m+n\to\infty}\left\{\sigma_{m,n}M\left[p_{m,n}^{(i)},r\right]\right\}^{\frac{1}{m+n}},$$

(2.4) $$\nu^{(i)}[0+]=\liminf_{m+n\to\infty}\left\{\sigma_{m,n}M\left[p_{m,n}^{(i)},r\right]\right\}^{\frac{1}{m+n}}$$

whenever

$$M\left[p_{m,n}^{(i)},r\right]=\max_{\bar{S}_r}|p_{m,n}^{(i)}(z,w)|.$$

Therefore the transposed sets $\{\tilde{p}_{m,n}^{(i)}(z,w)\}$; $i=1,2$ satisfy the conditions:

(2.5) $$\tilde{\mu}^{(i)}[0+]=0$$

(2.6) $$\tilde{\nu}^{(i)}[0+]\succ 0; \qquad \forall\, r \succ 0$$

where

(2.7) $$\tilde{\mu}^{(i)}[0+]=\limsup_{m+n\to\infty}\left\{\sigma_{m,n}M\left[\tilde{p}_{m,n}^{(i)},\frac{1}{r}\right]\right\}^{\frac{1}{m+n}},$$

(2.8) $$\tilde{\nu}^{(i)}[0+]=\liminf_{m+n\to\infty}\left\{\sigma_{m,n}M\left[\tilde{p}_{m,n}^{(i)},\frac{1}{r}\right]\right\}^{\frac{1}{m+n}}$$

and

$$M\left[\tilde{p}_{m,n}^{(i)},\frac{1}{r}\right]=\max_{\bar{S}_r}|\tilde{p}_{m,n}^{(i)}(z,w)|.$$

Also, the transposed inverse set $\{\hat{p}_{m,n}^{(1)}(z,w)\}$ satisfy the following conditions:

(2.9) $$\hat{\mu}^{(1)}[0+]=0$$

(2.10) $$\hat{\mu}^{(1)}[0+]=\limsup_{m+n\to\infty}\left\{\sigma_{m,n}M\left[\hat{p}_{m,n}^{(1)},\frac{1}{r}\right]\right\}^{\frac{1}{m+n}},$$

To study the effectiveness of similar transposed set of polynomials at the origin, we present at the beginning some lemmas that explain this experiment in preparation to prove this effectiveness.

Lemma (2.1):

Following set $\{p_j(z,w)\}$ satisfies the condition (2.1), then the transposed power set $\{{}^m\tilde{p}_j(z,w)\}$ satisfies the condition ${}^m\tilde{\mu}[0+]=0$.

Proof:

By transposed product set, write the square transposed set $\{{}^2\tilde{p}_j(z,w)\}$ where

$$\{{}^2\tilde{p}_j(z,w)\}=\{\tilde{p}_j(z,w)\}\{\tilde{p}_j(z,w)\}$$

then the square set is the transposed product set of two coincident sets, each of which satisfies the condition (2.5), then the set $\left\{{}^{2}\tilde{p}_{j}(z,w)\right\}$ satisfies condition ${}^{2}\tilde{\mu}[0+]=0$. Therefore the power set $\left\{{}^{m}\tilde{p}_{j}(z,w)\right\}$ satisfies condition ${}^{m}\tilde{\mu}[0+]=0$ and the set $\left\{{}^{m+1}\tilde{p}_{j}(z,w)\right\}$ is transposed product of two sets $\left\{\tilde{p}_{j}(z,w)\right\}$ and $\left\{{}^{m}\tilde{p}_{j}(z,w)\right\}$ as follows:

$$\left\{{}^{m+1}\tilde{p}_{j}(z,w)\right\}=\left\{\tilde{p}_{j}(z,w)\right\}\left\{{}^{m}\tilde{p}_{j}(z,w)\right\}$$

which satisfies respective conditions (2.5) and ${}^{m}\tilde{\mu}[0+]=0$, then the power set $\left\{{}^{m+1}\tilde{p}_{j}(z,w)\right\}$ satisfies condition ${}^{m+1}\tilde{\mu}[0+]=0$ and the proof of that this lemma follows, by induction.

Lemma (2.2):

Following transposed sets $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ satisfy the condition (2.5) and the set $\left\{p^{(1)}_{m,n}(z,w)\right\}$ is algebraic one, then the similar transposed set $\left\{u_{m,n}(z,w)\right\}$ holds:

(2.11) $$\mu[0+]=\limsup_{m+n\to\infty}\left\{\sigma_{m,n}\max_{\bar{S}_r}\left|u_{mn,}(z,w)\right|\right\}^{\frac{1}{m+n}}=0.$$

Proof:

Let the set $\left\{\tilde{p}^{(1)}_{mn,}(z,w)\right\}$ satisfies the condition (2.5) and by lemma (2.1), it follows that power set $\left\{{}^{j}\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ accords the condition ${}^{j}\tilde{\mu}^{(1)}_{m,n}[0+]=0$. Hence by (2.7) we get

(2.12) $$\sigma_{m,n}M\left[{}^{j}\widehat{p}^{(1)}_{m,n},r_1\right]\prec k_1 r_2^{m+n};\qquad m,n\geq 0,\quad j\geq 1.$$

If the set $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ is an algebraic set then we get

(2.13) $$\widehat{p}^{(i)h,k}_{m,n}=\gamma^{h,k}_{m,n}\sigma_{m,n}\alpha_0+\sum\nolimits_{j=1}^{t_i}\alpha_j^{t_1}\,{}^{j}\tilde{p}^{(i)h,k}_{m,n}$$

where $\alpha_0^{t_i}$; $i=1,2$ are constituent and $\left\{{}^{j}\tilde{p}^{(1)}_{m,n}(z,w)\right\}$; $j\geq 1$ is the j-th power of transposed set $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$.

By (2.12) and using Cauchy's inequality, the relation (2.13) is

(2.14) $$\left|\overline{p}^{(i)h,k}_{m,n}\right|\prec k_1\beta_1(t_1+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_6^{m+n}}{r_5^{h+k}}$$

where $\max\limits_{0\leq j\leq 1}|\alpha_j|$.

Suppose that $\left\{\tilde{p}^{(2)}_{m,n}(z,w)\right\}$ satisfies (2.5), we have

(2.15) $$\sigma_{m,n}M\left[\tilde{p}^{(2)}_{m,n},\frac{1}{r_5}\right]\prec k_1\frac{1}{r_4^{m+n}};\qquad m,n\geq 0.$$

By relations (2.14), (2.15) and Cauchy's inequality we get

(2.16) $$M[u_{m,n};\frac{1}{r_7}]=\max_{\bar{S}_{1/r_7}}|u_{m,n}(z,w)|$$

$$\le \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}||\hat{p}^{(1)s,t}_{i,j}|\frac{1}{r_7^{s+t}\sigma_{s,t}}$$

$$\le \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}||\bar{p}^{(1)i,j}_{s,t}|\frac{1}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}|\frac{\sigma_{i,j}}{\sigma_{s,t}}\frac{r_6^{s+t}}{r_5^{i+j}}\frac{1}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}|\frac{1}{\sigma_{i,j}r_5^{i+j}}\frac{\sigma_{i,j}r_6^{s+t}}{\sigma_{s,t}}\frac{\sigma_{i,j}}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|M[\tilde{p}^{(2)}_{h,k},\frac{1}{r_5}]\frac{\sigma_{i,j}r_6^{s+t}}{\sigma_{s,t}}\frac{\sigma_{i,j}}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|\frac{1}{r_4^{h+k}\sigma_{h,k}}\frac{\sigma_{i,j}r_6^{s+t}}{\sigma_{s,t}}\frac{\sigma_{i,j}}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2M[\tilde{p}^{(1)}_{h,k},\frac{1}{r_4}]\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|\frac{1}{r_4^{h+k}\sigma_{h,k}}\frac{\sigma_{i,j}r_6^{s+t}}{\sigma_{s,t}}\frac{\sigma_{i,j}}{r_7^{s+t}\sigma_{s,t}}$$

$$\prec k_1k_2M[\tilde{p}^{(1)}_{m,n},\frac{1}{r_4}]$$

from which, we obtain

$$\mu\left[\frac{1}{r_7}\right]=\operatorname{Lim}\sup_{m+n\to\infty}\left\{M[u_{m,n};\frac{1}{r_7}]\right\}^{\frac{1}{m+n}}\le \operatorname{Lim}\sup_{m+n\to\infty}\left\{k_1k_2M\left[\tilde{P}^{(1)}_{m,n},\frac{1}{r_4}\right]\right\}^{\frac{1}{m+n}}=\tilde{\mu}^{(1)}\left[\frac{1}{r_4}\right]$$

where $k_2=\beta_1(t_1+1)$.

If r_4 *and* r_7 are chosen near to r then we get

$$\mu[0+]\le\tilde{\mu}^{(1)}[0+]=0.$$

Therefore we obtain

$$\mu[0+]=0,$$

Now, we can use the lemmas above to prove the following theorem concerning the effectiveness of similar transposed set $\{u_{m,n}(z,w)\}$ of two complex variables at the origin:

Theorem (2.1):

Let $\{p^{(i)}_{m,n}(z,w)\};\ i=1,2$ be two algebraic sets satisfy condition (2.5), and then the similar transposed set $\{u_{m,n}(z,w)\}$ will be effective at the origin.

Proof:

Since each of two sets $\{p^{(i)}_{m,n}(z,w)\};\ i=1,2$ is algebraic condition (2.5) satisfied

(2.17) $$\left|\bar{p}^{(1)h,k}_{m,n}\right|\prec k_1\beta_1(t_1+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_4^{m+n}}{r_3^{h+k}}$$

(2.18) $$\left|\bar{p}^{(2)h,k}_{m,n}\right| \prec k_1\beta_2(t_2+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_2^{m+n}}{r_1^{h+k}}.$$

Also, the sets $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ satisfy the condition (2.11), then we get

$$\mu[0+]=0.$$

Therefore

(2.19) $$\sigma_{m,n}M[u_{m,n};\frac{1}{r_6}] \prec k_1\frac{1}{r_5^{m+n}}.$$

Inserting (2.16), (2.17). (2.18) and using Cauchy's inequality in Cannon sum of similar transposed set $\{u_{m,n}(z,w)\}$ we get:

$$\begin{aligned}
\Phi_{m,n}[\frac{1}{r_6}] &= \sigma_{m,n}\sum_{(s,t)=0}^{(m,n)} \left|\bar{u}^{s,t}_{m,n}\right| M[u_{s,t};\frac{1}{r_6}] \\
&\le \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\hat{p}^{(2)i,j}_{h,k}||\hat{p}^{(1)s,t}_{i,j}|\frac{1}{r_5^{s+t}\sigma_{s,t}} \\
&\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\bar{p}^{(2)h,k}_{i,j}||\bar{p}^{(1)i,j}_{s,t}|\frac{1}{r_5^{s+t}\sigma_{s,t}} \\
&\prec k_1k_2\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}||\bar{p}^{(2)h,k}_{i,j}|\frac{\sigma_{i,j}}{\sigma_{s,t}}\frac{r_4^{s+t}}{r_3^{i+j}}\frac{1}{r_5^{s+t}\sigma_{s,t}} \\
&\prec k_1k_2\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|\frac{\sigma_{h,k}}{\sigma_{h,k}r_1^{h+k}}\frac{\sigma_{h,k}}{\sigma_{i,j}}\frac{\sigma_{i,j}}{\sigma_{s,t}}\frac{r_2^{i+j}}{r_3^{i+j}}\frac{r_4^{s+t}}{r_5^{s+t}\sigma_{s,t}} \\
&\prec k_1k_2k_3\sigma_{m,n}M[\tilde{p}_{m,n};\frac{1}{r_1}]
\end{aligned}$$

where

$$k_2=\beta_1(t_1+1);$$
$$k_3=\beta_2(t_2+1).$$

Therefore the Cannon works as follows

$$\Phi[\frac{1}{r_6}] \le \Phi^{(1)}[\frac{1}{r_1}]$$

chosen r_1 *and* r_6 near to $0+$ we get

$$\Phi[0+]=0.$$

That is to say, the similar transposed set $\{u_{m,n}(z,w)\}$ was effective at origin.

Now we are going to take into account non-algebraic sets of polynomials of two complex variables, for this suggestion we will take the following two lemmas:

Lemma (2.3):

If the transposed sets $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ satisfy condition (2.5) and the set $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ is general set satisfies the condition (2.6), which is effective at the origin of C^2, then the similar transposed set $\{u_{m,n}(z,w)\}$ holds:

(2.20) $$\mu[0+]=0.$$

Proof:

From effectiveness of the general set $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ at the origin, that

(2.21) $$\tilde{\omega}^{(1)}[0+] = \underset{m+n\to\infty}{Lim\ \sup}\left\{M[\tilde{p}^{(1)}_{m,n};\frac{1}{r}]\right\}^{\frac{1}{m+n}} = 0.$$

where

$$\tilde{\omega}^{(1)}[0+] = \sigma_{m,n}\sup_{\alpha,\beta}\sup_{\bar{S}_{\frac{1}{r}}}\left|\sum_{(h,k)-\alpha}^{\beta}\widehat{p}^{(1)h,k}_{m,n}\,p^{(1)}_{h,k}(z,w)\right|$$

or

$$\tilde{\omega}^{(1)}_{m,n}[\frac{1}{r_3}] \prec k(\frac{1}{r_4})^{m+n}\ ;\quad m,n\geq 0.$$

By conditions (2.5) and (2.6); we have

(2.22) $$\sigma_{m,n}M\left[\tilde{p}^{(2)}_{m,n},\frac{1}{r_4}\right] \prec k(\frac{1}{r_5})^{m+n}$$

(2.23) $$\sigma_{m,n}M\left[\tilde{p}^{(1)}_{m,n},\frac{1}{r_3}\right] \succ k(\frac{1}{r_2})^{m+n};\quad m,n\geq 0.$$

By (2.21), (2.22), (2.23) and Cauchy's inequality it follows that:

(2.24) $$M[u_{m,n};\frac{1}{r_3}] = \max_{\bar{S}_{\frac{1}{r_3}}}\left|u_{m,n}(z,w)\right|$$

$$\leq \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}||\widehat{p}^{(1)s,t}_{i,j}|(\frac{1}{r_3})^{s+t}\frac{1}{\sigma_{s,t}}$$

$$\leq \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}||\widehat{p}^{(1)s,t}_{i,j}|\frac{M[\tilde{P}^{(1)}_{i,j},\frac{1}{r_3}]}{M[\tilde{P}^{(1)}_{i,j},\frac{1}{r_3}]}(\frac{1}{r_3})^{s+t}\frac{1}{\sigma_{s,t}}$$

$$\leq \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}|\frac{\tilde{\omega}^{(1)}_{i,j}[\frac{1}{r_3}]}{\sigma_{s,t}M[\tilde{P}^{(1)}_{s,t},\frac{1}{r_3}]}\frac{1}{\sigma_{i,j}}(\frac{1}{r_3})^{s+t}$$

$$\prec k_1\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}||\tilde{p}^{(2)i,j}_{h,k}|\frac{1}{r_4^{i+j}}\frac{1}{\sigma_{i,j}}(\frac{r_2}{r_3})^{s+t}$$

$$\prec k_1\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}|M[\tilde{P}^{(2)}_{h,k},\frac{1}{r_4}](\frac{r_2}{r_3})^{s+t}$$

$$\prec k_1\sum_{(h,k)=0}^{(m,n)}\sum_{(s,t)=0}^{(i,j)}|\tilde{p}^{(1)h,k}_{m,n}|\frac{1}{r_5^{h+k}}\frac{1}{\sigma_{h,k}}(\frac{r_2}{r_3})^{s+t}$$

$$\prec k_1 M[\tilde{P}^{(1)}_{m,n},\frac{1}{r_5}].$$

Taking $m+n\to\infty$ and keeping in mind that the set $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ satisfies (2.5), we get

$$\mu[0+] \leq \tilde{\mu}^{(1)}[0+] = 0.$$

Therefore we get

$$\mu[0+]=0.$$

Lemma (2.4):

The similar transposed set $\{u_{m,n}(z,w)\}$, satisfies the condition

(2.25) $\qquad \nu[r]\succ 0 \quad for \;\; r\succ 0$

whenever the transposed sets $\{\tilde{p}^{(i)}_{m,n}(z,w)\}$; $i=1,2$ satisfy the condition (2.5) and (2.6) and the set $\{\tilde{p}^{(2)}_{m,n}(z,w)\}$ is effective at the origin.

Proof:

The transposed sets $\{\tilde{p}^{(i)}_{m,n}(z,w)\}$; $i=1,2$ satisfy the condition (2.6) we get

(2.26) $\qquad \sigma_{m,n}M[\tilde{p}^{(1)}_{h,k};\frac{1}{r_4}]\prec k(\frac{1}{r_5})^{m+n}.$

Also, the transposed set $\{\tilde{p}^{(2)}_{m,n}(z,w)\}$ is effective at the origin, satisfies conditions (2.5) and (2.6), so must be its inverse transposed set $\{\hat{p}^{(2)}_{m,n}(z,w)\}$, thus we get

(2.27) $\qquad \sigma_{m,n}M[\hat{p}^{(2)}_{m,n};\frac{1}{r_3}]\prec k(\frac{1}{r_4})^{m+n}$

The similar transposed set $\{u_{m,n}(z,w)\}$ written in the form:

$$\{u_{m,n}(z,w)\}=\{\tilde{p}^{(1)}_{m,n}(z,w)\}\{\tilde{p}^{(2)}_{m,n}(z,w)\}\{\hat{p}^{(1)}_{m,n}(z,w)\}$$

Therefore

$$\{\tilde{p}^{(1)}_{m,n}(z,w)\}=\{u_{m,n}(z,w)\}\{\tilde{p}^{(1)}_{m,n}(z,w)\}\{\hat{p}^{(2)}_{m,n}(z,w)\}$$

from which we get

$$\tilde{p}^{(1)}_{m,n}(z,w)=\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|u^{h,k}_{m,n}||\tilde{p}^{(1)i,j}_{h,k}||\hat{p}^{(2)s,t}_{i,j}|z^s w^t.$$

By (2.26), (2.27) and Cauchy's inequality we obtain:

$$\begin{aligned} r^{m+n} &\prec \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|u^{h,k}_{m,n}||\tilde{p}^{(1)i,j}_{h,k}||\hat{p}^{(2)s,t}_{i,j}|\frac{1}{\sigma_{s,t}r_3^{s+t}} \\ &\prec \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)}|u^{h,k}_{m,n}||\tilde{p}^{(1)i,j}_{h,k}||\hat{p}^{(2)s,t}_{i,j}|\frac{1}{\sigma_{s,t}r_3^{s+t}} \\ &\prec \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}|u^{h,k}_{m,n}||\tilde{p}^{(1)i,j}_{h,k}|M[\hat{p}^{(2)}_{i,j};\frac{1}{r_3}] \\ &\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}|u^{h,k}_{m,n}||\tilde{p}^{(1)i,j}_{h,k}|\frac{1}{r_4^{i+j}\sigma_{i,j}} \\ &\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}|u^{h,k}_{m,n}|M[\tilde{p}^{(1)}_{h,k};\frac{1}{r_4}] \\ &\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}|u^{h,k}_{m,n}|\frac{1}{r_5^{h+k}\sigma_{h,k}}\prec k_1\sigma_{m,n}M[u_{m,n};\frac{1}{r_5}] \end{aligned}$$

so that

$$\nu\left[\frac{1}{r_5}\right] \geq r \succ 0.$$

Chosen r_5 near to r then we have

$$\nu[r] \succ 0 \quad for \;\; r \succ 0.$$

Theorem (2.3):

The general transposed set of polynomials of two complex variables $\left\{\tilde{p}^{(1)}_{m,n}(z,w)\right\}$ is effective at the origin of C^2, satisfies condition (2.5) and (2.6). Then the similar transposed set $\left\{u_{m,n}(z,w)\right\}$ is effective there, satisfies the conditions (2.20) and (2.25), if and only if, the transposed set $\left\{\tilde{p}^{(2)}_{m,n}(z,w)\right\}$ is effective at the origin of C^2 and satisfies the same conditions.

Proof:

From effectiveness of general sets $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ at the origin, we get

(2.28) $$\tilde{\omega}^{(1)}_{m,n}[\frac{1}{r_3}] \prec k(\frac{1}{r_4})^{m+n} \;\; ; \quad m,n \geq 0.$$

(2.29) $$\tilde{\omega}^{(2)}_{m,n}[\frac{1}{r_4}] \prec k(\frac{1}{r_5})^{m+n} \;\; ; \quad m,n \geq 0.$$

If the condition (2.20) of lemma (2.3) is satisfied then we get

(2.30) $$\sigma_{m,n} M\left[u_{m,n}, \frac{1}{r_2}\right] \prec k(\frac{1}{r_3})^{m+n}.$$

By condition (2.6); we obtain

(2.31) $$\sigma_{m,n} M\left[\tilde{p}^{(1)}_{m,n}, \frac{1}{r_3}\right] \succ k(\frac{1}{r_2})^{m+n}; \quad m,n \geq 0$$

(2.32) $$\sigma_{m,n} M\left[\tilde{p}^{(2)}_{m,n}, \frac{1}{r_4}\right] \succ k(\frac{1}{r_3})^{m+n}; \quad m,n \geq 0.$$

By (2.28) - (2.32) and Cauchy's inequality it follows that:

$$\Phi_{m,n}[\frac{1}{r_2}] = \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \left|\bar{u}^{h,k}_{m,n}\right| M[u_{h,k};\frac{1}{r_2}]$$

$$\leq \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|| \hat{p}^{(2)i,j}_{h,k} || \hat{p}^{(1)s,t}_{i,j}| M[u_{s,t};\frac{1}{r_2}]$$

$$\prec k_1 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|| \hat{p}^{(2)i,j}_{h,k} || \hat{p}^{(1)s,t}_{i,j}| \frac{1}{\sigma_{s,t} r_3^{s+t}}$$

$$\prec k_1 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}^{(1)h,k}_{m,n}|| \hat{p}^{(2)i,j}_{h,k} || \hat{p}^{(1)s,t}_{i,j}| \frac{\sigma_{s,t} M[\tilde{p}^{(1)}_{s,t};\frac{1}{r_3}]}{\sigma_{s,t} M[\tilde{p}^{(1)}_{s,t};\frac{1}{r_3}]} \frac{1}{\sigma_{s,t} r_3^{s+t}}$$

$$\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\hat{p}_{h,k}^{(2)i,j}| \frac{\tilde{\omega}_{i,j}^{(1)}[\frac{1}{r_3}]}{\sigma_{s,t}}\frac{r_2^{s+t}}{r_3^{s+t}}$$

$$\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\hat{p}_{h,k}^{(2)i,j}| \frac{\sigma_{i,j}M[\tilde{p}_{i,j}^{(2)};\frac{1}{r_4}]}{\sigma_{i,j}M[\tilde{p}_{i,j}^{(2)};\frac{1}{r_4}]}\frac{1}{r_4^{i+j}}\frac{r_2^{s+t}}{\sigma_{s,t}r_3^{s+t}}$$

$$\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}| \frac{\tilde{\omega}_{h,k}^{(2)}[\frac{1}{r_4}]}{\sigma_{s,t}}\left(\frac{r_3}{r_4}\right)^{i+j}\left(\frac{r_2}{r_3}\right)^{s+t}$$

$$\prec k_1\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}| \frac{1}{\sigma_{h,k}r_5^{h+k}}\left(\frac{r_3}{r_4}\right)^{i+j}\left(\frac{r_2}{r_3}\right)^{s+t}$$

$$\prec k_1\sigma_{m,n}M[\tilde{p}_{m,n}^{(1)};\frac{1}{r_5}].$$

Therefore

$$\Phi[0+]=0.$$

That to say the similar transposed set $\{u_{m,n}(z,w)\}$ effective at the origin, and satisfies conditions (2.20) and (2.25), by using lemmas (2) and (3) respectively and the "if" statement follows.

Now, to prove the only if, write

$$\{\tilde{p}_{m,n}^{(2)}(z,w)\}=\{\hat{p}_{m,n}^{(1)}(z,w)\}\{u_{m,n}(z,w)\}\{\tilde{p}_{m,n}^{(1)}(z,w)\}$$

let the set $\{u_{m,n}(z,w)\}$ is effective there satisfies conditions (2.20) and (2.25), the inverse transposed set $\{\hat{p}_{m,n}^{(1)}(z,w)\}$ is effective at the origin and satisfies the same conditions. Then the similar transposed set $\{\tilde{p}_{m,n}^{(2)}(z,w)\}$ is effective at the origin, and satisfies conditions (2.20) and (2.25).

3- Effectiveness of similar transposed set of polynomials in open hyperspheres

Now we are investigating the effectiveness of similar transposed set $\{u_{m,n}(z,w)\}$ of polynomials of two complex variables in open hyperspheres whenever the constituent sets are effective there.

We can only be algebraic and compliance with the relevant requirements

$$(3.1)\qquad \tilde{\mu}^{(i)}\left[\frac{1}{r}\right]=\operatorname*{Lim\,sup}_{m+n\to\infty}\left\{\sigma_{m,n}M[\tilde{p}_{m,n}^{(i)};\frac{1}{r}]\right\}^{\frac{1}{m+n}}\prec\frac{1}{R};\quad for\ all\ r\succ R\ ,\ i=1,2.$$

For this purpose, we give the following lemma:

Lemma (3.1)

The transposed set and the power transposed set accord to the same condition (3.1).

Proof:

First suppose that the set $\{\tilde{p}_{m,n}(z,w)\}$ satisfies condition

(3.2) $$\tilde{\mu}\left[\frac{1}{r}\right]=\underset{m+n\to\infty}{Lim\sup}\left\{\sigma_{m,n}M[\tilde{p}_{m,n};\frac{1}{r}]\right\}^{\frac{1}{m+n}}\prec\frac{1}{R};\quad for\ all\ r\succ R.$$

then

(3.3) $$\sigma_{m,n}M[\tilde{p}_{m,n};\frac{1}{r}]\prec k\ (\frac{1}{r_1})^{m+n};\quad (m,n)\geq 0.$$

The transposed product set of the two sets $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$, are satisfy the following relation:

$$^{2}\tilde{p}_{m,n}(z,w)=\sum_{(h,k)=0}^{(m,n)}\tilde{p}^{h,k}_{m,n}\tilde{p}_{h,k}(z,w).$$

By relation (3.3) and using Cauchy's inequality we get

$$\sigma_{m,n}M[^{2}\tilde{p}_{m,n};\frac{1}{r}]\leq\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}|\tilde{p}^{h,k}_{m,n}|M[\tilde{p}_{h,k};\frac{1}{r}]$$
$$\prec k_1\ \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}|\tilde{p}^{h,k}_{m,n}|\frac{1}{r_1^{h+k}\sigma_{h,k}}.$$
$$\prec k_1\ \sigma_{m,n}M[\tilde{p}_{m,n};\frac{1}{r_1}].$$

So that

$$^{2}\tilde{\mu}[\frac{1}{r}]\leq\tilde{\mu}[\frac{1}{r_2}]\prec\frac{1}{R};\quad for\ all\ r\succ R.$$

Also by the same way we can prove that

$$^{\lambda-1}\tilde{\mu}[\frac{1}{r_2}]\leq{}^{\lambda-2}\tilde{\mu}[\frac{1}{r_3}]\prec\frac{1}{R};\quad for\ all\ r\succ R$$

and by induction we get

$$^{\lambda}\tilde{\mu}[\frac{1}{r}]\prec\frac{1}{R};\quad for\ all\quad r\succ R.$$

We can deduce the following theorem from this Lemma:

Theorem (3.1):

When $\left\{\tilde{p}^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ two algebraic sets are effective and satisfy in the open hyperspheres $S_{1/r}$ the condition (3.2), then the similar transposed set $\left\{u_{m,n}(z,w)\right\}$ is effective in open hyperspheres $S_{1/R}$.

Proof:

Let two sets $\left\{p^{(i)}_{m,n}(z,w)\right\}$; $i=1,2$ be algebraic sets each fulfilling the following conditions:

(3.4) $$\left|\overline{p}^{(1)h,k}_{m,n}\right|\prec k_1\beta_1(t_1+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_3^{m+n}}{r_2^{h+k}}$$

(3.5) $$\sigma_{m,n}M[\tilde{p}^{(2)}_{i,j};\frac{1}{r_2}]\prec k_1(\frac{1}{r_3})^{m+n}.$$

Therefore

(3.6)
$$M[u_{m,n},\frac{1}{r_4}]=\max_{S_{r_4}}|u_{m,n}(z,w)|$$
$$\leq \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\tilde{p}_{h,k}^{(2)i,j}||\widehat{p}_{i,j}^{(1)s,t}|\frac{1}{r_4^{s+t}\sigma_{s,t}}$$
$$\leq \sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\tilde{p}_{h,k}^{(2)i,j}||\overline{p}_{s,t}^{(1)i,j}|\frac{1}{r_4^{s+t}\sigma_{s,t}}$$
$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\tilde{p}_{h,k}^{(2)i,j}|\frac{\sigma_{i,j}}{\sigma_{s,t}}\frac{r_3^{s+t}}{r_2^{i+j}}\frac{1}{r_4^{s+t}\sigma_{s,t}}$$
$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\tilde{p}_{h,k}^{(2)i,j}|\frac{1}{r_2^{i+j}}\frac{\sigma_{i,j}}{\sigma_{s,t}}\frac{r_3^{s+t}}{r_4^{s+t}\sigma_{s,t}}$$
$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}|M[\tilde{p}_{h,k}^{(2)},\frac{1}{r_2}]\frac{\sigma_{i,j}\sigma_{i,j}}{\sigma_{s,t}}\frac{r_3^{s+t}}{r_4^{s+t}\sigma_{s,t}}$$
$$\prec k_1k_2\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}|\frac{1}{r_3^{h+k}\sigma_{h,k}}\frac{\sigma_{i,j}\sigma_{i,j}}{\sigma_{s,t}}\frac{r_3^{s+t}}{r_4^{s+t}\sigma_{s,t}}$$
$$\prec k_1k_2M[\tilde{p}_{m,n}^{(1)},\frac{1}{r_3}]$$

where
$$k_2=\beta_1(t_1+1).$$
Thus
$$\mu[\frac{1}{r_4}]\leq \limsup_{m+n\to\infty}\left\{\sigma_{m,n}M[u_{m,n},\frac{1}{r_3}]\right\}^{\frac{1}{m+n}}$$
$$\leq \limsup_{m+n\to\infty}\left\{\sigma_{m,n}k_1k_2M[\tilde{p}_{m,n}^{(1)};\frac{1}{r_3}]\right\}^{\frac{1}{m+n}}\leq \tilde{\mu}^{(1)}[\frac{1}{r_3}]$$
and
$$\sigma_{m,n}M[u_{m,n},\frac{1}{r_4}]\prec k_1(\frac{1}{r_5})^{m+n}.$$
Now, taking

(3.7)
$$\left|\overline{p}_{m,n}^{(1)h,k}\right|\prec k_1\beta_1(t_1+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_4^{m+n}}{r_3^{h+k}}$$

(3.8)
$$\left|\overline{p}_{m,n}^{(2)h,k}\right|\prec k_1\beta_2(t_2+1)\frac{\sigma_{h,k}}{\sigma_{m,n}}\frac{r_2^{m+n}}{r_1^{h+k}}.$$

By (3.6), (3.7), (3.8) and Cauchy's inequality it follows that:
$$\Phi_{m,n}[\frac{1}{r_4}]=\sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\left|\overline{u}_{m,n}^{h,k}\right|M[u_{h,k};\frac{1}{r_4}]$$
$$\leq \sigma_{m,n}\sum_{(h,k)=0}^{(m,n)}\sum_{(i,j)=0}^{(h,k)}\sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\tilde{p}_{h,k}^{(2)i,j}||\widehat{p}_{i,j}^{(1)s,t}|M[u_{s,t};\frac{1}{r_4}]$$

$$\leq \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\bar{p}_{i,j}^{(2)h,k}||\bar{p}_{s,t}^{(1)i,j}| M[u_{s,t};\frac{1}{r_4}]$$

$$\prec k_1 k_2 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\bar{p}_{i,j}^{(2)h,k}||\bar{p}_{s,t}^{(1)i,j}| \frac{1}{r_5^{s+t}\sigma_{s,t}}$$

$$\prec k_1 k_2 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}||\bar{p}_{i,j}^{(2)h,k}| \frac{\sigma_{i,j}}{\sigma_{s,t}} \frac{r_4^{s+t}}{r_3^{i+j}} \frac{1}{r_5^{s+t}\sigma_{s,t}}$$

$$\prec k_1 k_2 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}| \frac{\sigma_{h,k}}{\sigma_{i,j}} \frac{r_2^{i+j}}{r_1^{h+k}} \frac{\sigma_{i,j}}{\sigma_{s,t}} \frac{r_4^{s+t}}{r_3^{i+j}} \frac{1}{r_5^{s+t}\sigma_{s,t}}$$

$$\prec k_1 k_2 \sigma_{m,n} \sum_{(h,k)=0}^{(m,n)} \sum_{(i,j)=0}^{(h,k)} \sum_{(s,t)=0}^{(i,j)} |\tilde{p}_{m,n}^{(1)h,k}| \frac{\sigma_{h,k}}{\sigma_{h,k} r_1^{h+k}} \frac{\sigma_{h,k}}{\sigma_{i,j}} \frac{r_2^{i+j}}{r_1^{h+k}} \frac{\sigma_{i,j}}{\sigma_{s,t}} \frac{r_4^{s+t}}{r_3^{i+j}} \frac{1}{r_5^{s+t}\sigma_{s,t}}$$

$$\prec k_1 k_2 k_3 \sigma_{m,n} M[\tilde{p}_{m,n}^{(1)};\frac{1}{r_1}]$$

where

$$k_2 = \beta_1(t_1+1); \qquad k_3 = \beta_2(t_2+1).$$

So that

$$\mu[\frac{1}{r_4}] \leq \limsup_{m+n\to\infty} \left\{\Phi_{m,n}[\frac{1}{r_4}]\right\}^{\frac{1}{m+n}} \leq \tilde{\mu}^{(1)}[\frac{1}{r_1}] \prec \frac{1}{R}; \quad \textit{for all} \quad r \succ R.$$

In other words, the similar transposed set $\{u_{m,n}(z,w)\}$ effective established an open hypersphere $S_{1/R}$.

4- Effectiveness of similar transposed set of polynomials in closed hyperspheres

Now we are giving some important results for the effectiveness of similar transposed sets of polynomials in some other regions, and a new study of the effectiveness of these polynomials is being considered, and proof of these results is given in a similar manner to those previously identified in this study.

First effectiveness of transposed basic set of polynomials of two complex variables $\{\tilde{p}_{m,n}(z,w)\}$ in closed hyperspheres $\bar{S}_{\frac{1}{r}}$ whenever the simple basic set $\{p_{m,n}(z,w)\}$ effective in closed hyperspheres $\bar{S}_r$ with leading coefficients unity.

The effectiveness of transposed basic set of polynomials given in following theorem:

Lemma (4.1):

Suppose that $\{p_{m,n}(z,w)\}$ be simple monic set with leading coefficients unity effective in the hyperspheres $\bar{S}_r$, then the transposed set $\{\tilde{p}_{m,n}(z,w)\}$ is effective in the closed hyperspheres $\bar{S}_{\frac{1}{r}}$.

Let $\{\hat{p}_{m,n}(z,w)\}$ be a transposed inverse set of polynomials of the set $\{p_{m,n}(z,w)\}$ where

$$\hat{p}_{m,n}(z,w) = \sum_{(h,k)=0}^{(m,n)} \hat{p}_{m,n}^{h,k} z^h w^k = \sum_{(h,k)=0}^{(m,n)} \bar{p}_{h,k}^{m,n} z^h w^k.$$

Effectiveness the transposed inverse set $\{\hat{p}_{m,n}(z,w)\}$ of polynomials in closed hyperspheres $\bar{S}_{\frac{1}{r}}$; $r > 0$ with leading coefficients unity, whenever the basic set $\{p_{m,n}(z,w)\}$ is effective in the same region under the same condition as follows:

Lemma (4.2):

Suppose that $\{p_{m,n}(z,w)\}$ be simple monic set with leading coefficients unity effective in the hyperspheres $\bar{S}_r$, then the transposed inverse set $\{\hat{p}_{m,n}(z,w)\}$ is effective in the closed hyperspheres $\bar{S}_{\frac{1}{r}}$.

Let $\{p^{(i)}_{m,n}(z,w)\}$; $i = 1,2$ are basic sets of polynomials of two complex variables and the set $\{\tilde{q}_{m,n}(z,w)\}$ is called the product set of the two sets $\{\tilde{p}^{(i)}_{m,n}(z,w)\}$; $i = 1,2$,

$$\{\tilde{q}_{m,n}(z,w)\} = \{\tilde{p}^{(1)}_{m,n}(z,w)\}\{\tilde{p}^{(2)}_{m,n}(z,w)\} ,$$

Now, the effectiveness of transposed product basic set of polynomials of two complex variables $\{\tilde{q}_{m,n}(z,w)\} = \{\tilde{p}^{(1)}_{m,n}(z,w)\}\{\tilde{p}^{(2)}_{m,n}(z,w)\}$ in closed hyperspheres $\bar{S}_{\frac{1}{r}}$ whenever the sets $\{p^{(i)}_{m,n}(z,w)\}$; $i = 1,2$ are simple monic ones, are effective in closed hyperspheres $\bar{S}_r$ with leading coefficients unity, i.e. $p^{(i)m,n}_{m,n} = 1$; $i = 1,2$.

Lemma (4.3):

Suppose that $\{p^{(i)}_{m,n}(z,w)\}$; $i = 1,2$ be simple monic set with leading coefficients unity effective in the hyperspheres $\bar{S}_r$, then the transposed product set $\{\tilde{q}_{m,n}(z,w)\}$ is effective in the closed hyperspheres $\bar{S}_{\frac{1}{r}}$.

Now, we give effectiveness of similar transposed set $\{u_{m,n}(z,w)\}$ in closed hyperspheres $\bar{S}_{\frac{1}{r}}$ whenever the transposed simple basic sets $\{\tilde{p}^{(i)}_{m,n}(z,w)\}$; $i = 1,2$ are effective in closed hyperspheres $\bar{S}_{\frac{1}{r}}$ and also, whenever the simple basic sets $\{p^{(i)}_{m,n}(z,w)\}$ are effective in open hyperspheres $\bar{S}_r$ with leading coefficients unity, $p^{(i)m,n}_{m,n} = 1$; $i = 1,2$.

Theorem (4.1):

Suppose that $\{p^{(i)}_{m,n}(z,w)\}$; $i = 1,2$ be two simple monic sets with leading coefficients unity effective in the hyperspheres $\bar{S}_r$, then the similar transposed set $\{u_{m,n}(z,w)\}$ is effective in the closed hyperspheres $\bar{S}_{\frac{1}{r}}$.

The inverse set of polynomials of two complex variables $\{\bar{u}_{m,n}(z,w)\}$ as follows:

$$\{\bar{u}_{m,n}(z,w)\} = \{\tilde{p}^{(1)}_{m,n}(z,w)\}\{\tilde{p}^{(2)}_{m,n}(z,w)\}\{\hat{p}^{(1)}_{m,n}(z,w)\}$$

Now, we give effectiveness of inverse similar transposed set $\{\bar{u}_{m,n}(z,w)\}$ in closed hyperspheres $\bar{S}_{\frac{1}{r}}$, whenever the simple basic sets $\{p^{(i)}_{m,n}(z,w)\}$; $i=1,2$ are effective in closed hyperspheres $\bar{S}_r$ with leading coefficients unity, i.e. $p^{(i)m,n}_{m,n}=1$; $i=1,2$.

Theorem (4.2):

Suppose that $\{p^{(i)}_{m,n}(z,w)\}$; $i=1,2$ be two simple monic sets with leading coefficients unity effective in the hypersphere $\bar{S}_r$, then the inverse similar transposed set $\{\bar{u}_{m,n}(z,w)\}$ is effective in the closed hypersphere $\bar{S}_{\frac{1}{r}}$ if and only if the set $\{\tilde{p}^{(2)}_{m,n}(z,w)\}$ is effective there.

Conclusions

In this paper, where the correctness of the corresponding functions is effectiveness in origin, in the open hyperspheres and in the closed hyperspheres, a new comparison is proposed to study some significant properties of some corresponding functions in two complex variables, and this study is called a new one of its kinds. Generalization of the corresponding position and it has relevance in many areas of application and physics.

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