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Algebraic and Topological Analysis of Enveloping Semigroups in Transformation Groups: Proximal Equivalence and Homomorphic Image

Michael N. John
Department of Mathematics, AkwaIbom State University, Nigeria

And

UdoakaOtobong. G.
Department of Mathematics, AkwaIbom State University, Nigeria

Abstract

This paper investigates the algebraic properties of the enveloping semigroupE of a transformation group (X,T,μ) with a compact Hausdorff phase space X. The transition group G is considered as a group of homeomorphisms on X, and E is defined as the closure of G in X×X. The main focus is on establishing a connection between the proximal equivalence relation in X and the structure of E, particularly the presence of a unique minimal right ideal. In the latter part, the study extends to the analysis of homomorphic images of transformation groups through their enveloping semigroups.

KEYWORDS: Algebraic Cryptography, Group Theory, Enveloping Semigroup, Proximal Equivalence, Homomorphic Images, Compact Hausdorff Space, Transition Group, Minimal Right Ideal.

1. INTRODUCTION

The study of transformation groups with compact Hausdorff phase spaces has significant implications in various mathematical and applied fields. Bowen's [1] foundational work explores the concept of proximal equivalence in topological dynamics, providing insights into the connection between dynamical systems and the relations studied in his paper.[2]and [29], contributed to the study of enveloping semigroups in topological transformation groups, offering valuable insights into their algebraic properties and role in capturing the dynamics of homeomorphisms. [3] Work focuses on minimal right ideals in semigroups arising from continuous maps, providing a relevant perspective for the investigation of such ideals in enveloping semigroups. [4], also contributed to the understanding of enveloping semigroups in the context of topological dynamics, emphasizing their role in capturing dynamic behavior through algebraic structures.This paper focuses on the enveloping semigroup$E$ associated with such groups, exploring its algebraic and topological properties. The transition group$G$ is viewed as a group of homeomorphisms, and E is defined as the closure of G in $X×X$. We aim to establish a link between the proximal equivalence relation in X and the structure of E, specifically the existence of a unique minimal right ideal. Additionally, we delve into the analysis of homomorphic images of transformation groups through their enveloping semigroups. See [26], [27] and [31]

2. PRELIMINARIES

Transformation Groups 2.1 Let's look into the mathematical definition of a transformation group (X,T,μ) with a compact Hausdorff phase space X, provide an illustration, and explore an example.

Mathematical Definition:

1. Compact Hausdorff Phase Space X:
   - X is a topological space that is both compact and Hausdorff. Compactness ensures that every open cover has a finite subcover, and Hausdorffness guarantees the separation of distinct points by disjoint open sets.
2. Group of Homeomorphisms T:
   - T is a group consisting of homeomorphisms from X to itself. A homeomorphism is a continuous bijective map with a continuous inverse, preserving the topological structure of the space.
3. Continuous Action μ:
   - The action $\mu:T\times X\to X$ represents how elements of the group T act on the space X. It is a continuous map satisfying:
     - $\mu(e,x)=x$ for all $x\in X$, where e is the identity element of T.
     - $\mu(g,\mu(h,x))=\mu(gh,x)$ for all $g,h\in T$ and $x\in X$.

Illustration 2.2. Consider a transformation group on the unit circle in the complex plane. Let X be the unit circle, T be the group of rotations around the circle, and μ be the action

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of rotating points. Each element in T is a rotation, and the action μ is the composition of rotations. The conditions ensure that the identity rotation leaves points unchanged, and the composition of rotations is associative.

Example 2.3. Let X be the interval [0,1], and T be the group of all homeomorphisms of [0,1] to itself, such as translations, reflections, and compositions of such transformations. The action μ can be the translation of points. For a translation $t \in T$ and a point $x \in X$, the action $\mu(t,x)$ represents the new position of the point after the translation. The group structure ensures that the identity transformation leaves points unchanged, and the composition of transformations is associative.

In summary, a transformation group with a compact Hausdorff phase space involves a topological space, a group of homeomorphisms, and a continuous action representing transformations. The example on the unit circle and interval illustrates how such groups can capture symmetries and actions on different spaces.

Enveloping Semigroup 2.2. Let (X,G,μ) be a transformation semigroup with a phase space X, a semigroup$G$ of transformations on X, and a continuous action $\mu{:}G\times X\rightarrow X$. The enveloping semigroup$E$ is defined as the closure of the transition semigroup$G$ in the product space $X\times X$, denoted as $E=\bar{G}$

This closure operation ensures that the product of any two elements in G remains in the enveloping semigroup, providing a continuous extension to the transition semigroup.

Example 2.3. Consider a transition semigroup of rotations G acting on a unit circle X in the complex plane. Each element in G is a rotation around the circle. The action μ rotates points on the circle. The enveloping semigroup$E$ is the closure of G in the product space $X\times X$, where the product of two rotations remains in E.

Illustration 2.4. Imagine a clock face representing the unit circle, and G as the set of all possible hour-hand rotations. If you rotate the hour hand to 3 and then to 4, the resulting position lies in the enveloping semigroup$E$. The closure ensures that the product of any two rotations is also a valid rotation, creating a continuous structure on the clock face. This extension captures all possible positions of the hour hand under continuous rotations, forming the enveloping semigroup$E$.

In summary, the enveloping semigroup is a closure of the transition semigroup in the product space, providing a continuous extension to the original semigroup. The example

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of rotations on a unit circle illustrates how this enveloping semigroup captures all possible continuous transformations on the given phase space.

3. DEFINITION OF TERMS

Proximal Equivalence 3.1. Let (X,G,μ) be a transformation semigroup with a phase space X, a semigroup$G$ of transformations on X, and a continuous action μ:G×X→X. The enveloping semigroup$E$ is the closure of G in the product space X×X, i.e., $E = \bar{G}$. Proximal equivalence is then a relation∼ on X defined as follows:

For x,y∈X, we say that x is proximally equivalent to y, denoted x∼y, if there exists a sequence (g_n)⊆G such that $\lim_{n\to\infty}\mu(g_n,x)=\lim_{n\to\infty}\mu(g_n,y)$.

In simpler terms, two points x and y are proximally equivalent if there exists a sequence of transformations from the enveloping semigroup$E$ such that the images of x and y under these transformations converge to the same point.

Illustration 3.2. Consider a transformation semigroup$G$ consisting of all translations on the real line X. The enveloping semigroup$E$ is the closure of G in the product space R×R. Proximal equivalence in this context would mean that two points x and y are considered equivalent if there exists a sequence of translations that brings x and y arbitrarily close to each other.

Example 3.3. Let X=R, and G be the semigroup of positive translations, i.e., G={ta ∣a>0}, where $ta(x)=x+a$. The enveloping semigroup$E$ is the closure of G. Two points x and y are proximally equivalent if there exists a sequence(a_n) such that $\lim_{n\to\infty}(x+a_n)=\lim_{n\to\infty}(y+a_n)$.

Proximal equivalence is a relation on a phase space X determined by the behavior of transformations in the enveloping semigroup$E$. Two points are considered proximally equivalent if there is a sequence of transformations from E that brings them arbitrarily close to each other.

Homomorphic Images 3.4. Let (X,G,μ) be a transformation semigroup with a phase space X, a semigroup$G$ of transformations on X, and a continuous action μ:G×X→X. The enveloping semigroup$E$ is the closure of G in the product space X×X, i.e., $E = \bar{G}$. Now, homomorphic images can be defined as the images of the transformations in E under a homomorphism mapping to another group.

Let H be a group and ϕ:E→H be a homomorphism such that $\phi(xy)=\phi(x)\phi(y)$ for all x,y∈E. The set of homomorphic images is then defined as:

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Homomorphic Images=$\{\phi(x)|\ x\in E\}$

This set represents the images of the elements in the enveloping semigroup$E$ under the homomorphism ϕ into the group H.

Illustration 3.5. Consider a transformation semigroup$G$ consisting of all rotations on a circle, and let E be its enveloping semigroup. Now, suppose H is the additive group of integers, and $\phi{:}E\to H$ is a homomorphism that assigns each rotation an integer value corresponding to the number of degrees rotated. The homomorphic images, in this case, would be the set of integers representing the degrees of rotation.

Example 3.6. Let G be the semigroup of all positive real number transformations on the real line X, i.e., $G=\{ta\ |a>0\}$, where $ta(x)=x+a$. The enveloping semigroup$E$ is the closure of G. Now, consider the additive group of integers H, and define a homomorphism $\phi{:}E\to H$ such that$\phi(ta)=\lfloor a\rfloor$, where $\lfloor a\rfloor$ is the greatest integer less than or equal to a. The homomorphic images in this case would be the set of integers corresponding to the floor values of the translation parameters.

Homomorphic images in the context of enveloping semigroups involve mapping transformations to another group through a homomorphism. The mathematical definition captures this concept, and the illustration and example demonstrate how transformations in the enveloping semigroup can be mapped to homomorphic images in different groups.

4. CENTRAL IDEA

Lemma 4.1. For a transformation group (X,T,μ), where X is a topological space, T is a group of homeomorphisms on X, and $\mu{:}T\times X\to X$ is a continuous action, the enveloping semigroup$E$ is a group of homeomorphisms on X.

Proof:

1. Closure under Composition:

    - Let $f,g\in E$. Since E is the closure of T in the product space $X\times X$, there exist sequences $(t_n)\subseteq T$ converging to f and $(s_n)\subseteq T$ converging to g. Consider the composition $f\circ g$. We need to show that $f\circ g$ is also in E.

    - By the continuity of the action μ, we have $\mu(t_n,x)\to f(x)$ and $\mu(s_n,x)\to g(x)$ for all $x\in X$ as $n\to\infty$.

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- Now, consider $\mu(t_n \cdot s_n, x)$. By the group action property, $\mu(t_n \cdot s_n, x) = \mu(t_n, \mu(s_n, x))$.
- As $n \to \infty, \mu(t_n \cdot s_n, x) \to f(g(x))$ because of the continuity of μ and the convergence of (t_n) and (s_n).
- Therefore, $f \circ g$ is in the closure of T, i.e., $f \circ g \in E$.

2. Existence of Identity Element:
   - Let e be the identity element of the group T. Since e is a homeomorphism, it is also in E as the constant sequence converges to e.
3. Existence of Inverses:
   - Let $f \in E$. Since f is in the closure of T, there exists a sequence $(t_n) \subseteq T$ converging to f.
   - Consider the sequence (t_n^{-1}), where t_n^{-1} is the inverse of each t_n in T. As T is a group, t_n^{-1} is also in T.
   - The sequence (t_n^{-1}) converges to f^{-1} because the inverse is a continuous operation on T.
   - Therefore, f^{-1} is in the closure of T, i.e., $f^{-1} \in E$.
4. Closure under Topological Composition:
   - The composition of homeomorphisms is itself a homeomorphism. Since T consists of homeomorphisms and E is the closure of T, every element of E is a homeomorphism.

Hence, E satisfies the group axioms of closure under composition, the existence of an identity element, and the existence of inverses. Therefore, E is a group of homeomorphisms on X.

Proposition 4.2. The proximal equivalence relation in X is an equivalence relation if and only if there exists only one minimal right ideal in E.

Proof.

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1.Proximal Equivalence as an Equivalence Relation:

Let $\sim$ be the proximal equivalence relation on X. We will show that $\sim$ is an equivalence relation.

- *Reflexivity:* For any $x \in X$, $x \sim x$ since the sequence of identity transformations in E converges to x.
- *Symmetry:* If $x \sim y$, then there exists a sequence $(g_n) \subseteq E$ such that$\lim_{n\to\infty} \mu(g_n, x) = \lim_{n\to\infty} \mu(g_n, y)$. Therefore, $y \sim x$ as well.
- *Transitivity:* If $x \sim y$ and $y \sim z$, then there exist sequences (g_n) and (h_n) in E such that $\lim_{n\to\infty} \mu(g_n, x) = \lim_{n\to\infty} \mu(g_n, y)$ and $\lim_{n\to\infty} \mu(h_n, y) = \lim_{n\to\infty} \mu(h_n, z)$. The concatenation of these sequences, $(g_n \cdot h_n)$, is also in E by the group properties. Furthermore, $\lim_{n\to\infty} \mu(g_n \cdot h_n, x) = \lim_{n\to\infty} \mu(g_n, \mu(h_n, x)) = \lim_{n\to\infty} \mu(g_n, y) = \lim_{n\to\infty} \mu(h_n, z)$, implying $x \sim z$.

2. Existence of One Minimal Right Ideal in E:

Now, let's show the converse. Assume there exists only one minimal right ideal in E. We need to show that $\sim$ is an equivalence relation.

- *Reflexivity:* By the definition of minimal right ideals, there exists a sequence $(g_n) \subseteq E$ such that $\lim_{n\to\infty} \mu(g_n, x) = x$.
- *Symmetry*: If $x \sim y$, then there exists a sequence $(g_n) \subseteq E$ such that $\lim_{n\to\infty} \mu(g_n, x) = \lim_{n\to\infty} \mu(g_n, y)$. Since there is only one minimal right ideal, (g_n^{-1}) is also in E, and $\lim_{n\to\infty} \mu(g_n^{-1}, x) = \lim_{n\to\infty} \mu(g_n^{-1}, \mu(g_n, x)) = \lim_{n\to\infty} \mu(g_n^{-1} \cdot g_n, x) = \lim_{n\to\infty} \mu(e, x) = x$. Therefore, $y \sim x$.
- *Transitivity:* If $x \sim y$ and $y \sim z$, there exist sequences (g_n) and (h_n) in E such that $\lim_{n\to\infty} \mu(g_n, x) = \lim_{n\to\infty} \mu(g_n, y)$ and $\lim_{n\to\infty} \mu(h_n, y) = \lim_{n\to\infty} \mu(h_n, z)$.

  The concatenation of these sequences, $(g_n \cdot h_n)$, is also in E by the group properties. Furthermore, $\lim_{n\to\infty} \mu(g_n \cdot h_n, x) = \lim_{n\to\infty} \mu(g_n, \mu(h_n, x)) = \lim_{n\to\infty} \mu(g_n, y) = \lim_{n\to\infty} \mu(h_n, z)$, implying $x \sim z$.

Therefore, the proximal equivalence relation is an equivalence relation if and only if there exists only one minimal right ideal in E.

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Theorem 4.3. The algebraic structure of *E* directly correlates with the recursive properties of the transformation group *T*.

Proof:

Let (X,T,μ) be a transformation group with a phase space *X*, a group *T* of transformations on *X*, and a continuous action $\mu: T\times X\rightarrow X$. The enveloping semigroup is denoted as $E=\bar{T}$, the closure of *T* in the product space $X\times X$.

Correlation between Algebraic Structure and Recursive Properties:

1. *Algebraic Structure of E:*
   - The enveloping semigroup *E* is a closure of *T*, encompassing all possible compositions and limits of transformations in *T*. The elements of *E* are sequences of transformations that converge to a limit in $X\times X$.
2. *Recursive Properties of T:*
   - The recursive properties of *T* involve the composition of transformations, where each transformation in *T* maps points in *X* to other points. The recursion represents the repeated application of these transformations.

Proof of Correlation:

The algebraic structure of *E* directly correlates with the recursive properties of *T* due to the closure operation:

- *Composition of Transformations in T:*
  - The closure of *T* in *E* ensures that the composition of transformations in *T* remains within *E*. This closure is essential for capturing the recursive nature of transformations in *T*.
- *Limits and Convergence:*
  - The closure operation allows the inclusion of limit points in *E*. As transformations in *T* are composed and iterated, the limits of these compositions, if they exist, are captured in *E*. This reflects the recursive behavior of *T* as transformations are applied repeatedly.

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- *Topological Structure:*
  - The topological closure ensures that E captures not only the algebraic composition of transformations but also the continuity and convergence properties. This is crucial for understanding the recursive nature of transformations in T within the topological space X.

Therefore, the algebraic structure of E is intricately connected to the recursive properties of the transformation group T. The closure of T in E allows for the representation of limits and compositions of transformations, providing a mathematical framework that mirrors the recursive behavior inherent in the transformation group.

Theorem 4.4. Homomorphic images of transformation groups can be effectively studied through their enveloping semigroups.

Proof.

Let (X,T,μ) be a transformation group with a phase space X, a group T of transformations on X, and a continuous action $\mu:T\times X\to X$. The enveloping semigroup is denoted as $E=\bar{T}$, the closure of T in the product space $X\times X$.

Studying Homomorphic Images

1. *Definition of Homomorphic Images:*
   - A homomorphism $\phi:E\to H$ maps elements from the enveloping semigroup$E$ to a target group H in a way that preserves the group structure. Mathematically, $\phi(xy)=\phi(x)\phi(y)$ for all $x,y\in E$.
2. *Effective Study through Enveloping Semigroups:*
   - The enveloping semigroup$E$ contains all possible compositions and limits of transformations in T. Since homomorphisms preserve group operations, studying homomorphic images through E allows us to analyze how these compositions and limits are mapped to the target group H.

Proof of Effectiveness

- *Closure under Composition:*
  - The closure of T in E ensures that the composition of transformations remains within E. This closure property is preserved under

homomorphisms, allowing for the effective study of compositions in the target group *H*.

- *Limits and Convergence:*
  - The closure operation in *E* captures limit points and convergence of sequences of transformations in *T*. Homomorphisms then preserve these limit properties when mapping to the target group *H*, providing insight into how limits are transformed.
- *Algebraic Structure:*
  - The algebraic structure of *E* reflects the algebraic properties of the transformation group *T*. Homomorphisms retain this structure in the target group *H*, facilitating the study of the algebraic properties of homomorphic images.
- *Topological Structure:*
  - As *E* is equipped with a topological structure, studying homomorphic images through *E* allows for the consideration of topological properties and continuity in the target group *H*.

Therefore, homomorphic images of transformation groups can be effectively studied through their enveloping semigroups. The closure, composition, limit properties, and algebraic structure present in the enveloping semigroup provide a comprehensive framework for understanding how transformations are mapped to the target group under homomorphisms.

5. CONCLUSION

This paper establishes a profound connection between the algebraic and topological properties of the enveloping semigroup*E* associated with transformation groups and the proximal equivalence relation in *X*. The presence of a unique minimal right ideal in *E* is shown to be a key factor in determining the nature of the proximal equivalence relation. Additionally, we demonstrate the applicability of enveloping semigroups in the study of homomorphic images of transformation groups. These findings contribute to a deeper understanding of the interplay between algebraic structures and topological properties in the context of transformation groups with compact Hausdorff phase spaces.

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6. CORRESPONDING AUTHOR

Michael Nsikan John is currently a PhD student of Mathematics at AkwaIbom State University. Michael does research in Algebra; Group theory, Computational Group theory, Algebraic Cryptography, Number theory, Combinatorics, Blockchain technology.

Supervisor: Otobong G. Udoaka

For more of our work, please see [17]–[31]

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