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Para-G Relations and Hirsch Length in Residually Nilpotent Groups

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Abstract

This research explores the interplay between residually nilpotent groups G and H, focusing on their relationship through the lens of para-G conditions and the Hirsch length. We establish criteria for H to be para-G concerning monomorphisms inducing isomorphisms between corresponding lower central quotients of G and H. Specifically, we investigate these conditions in the context of finitely generated residually nilpotent groups. Further, for certain polycyclic groups, we establish connections between para-G relations and the equality of Hirsch lengths. Additionally, we delve into the pro-nilpotent completions of these polycyclic groups, demonstrating their local polycyclic nature.

KEYWORDS: Residually Nilpotent Groups, Para-G Relations, Hirsch Length, Lower Central Quotients, Pro-Nilpotent Completions, Polycyclic Groups.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
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1. INTRODUCTION

Residually nilpotent groups play a pivotal role in group theory, and understanding their relationships is essential for exploring the underlying algebraic structures. The study by [1] provides foundational insights into para-G conditions in group theory, particularly in the context of residually nilpotent groups. Hall's work lays the groundwork for understanding the interconnections between groups and the criteria for para-G relations.The concept of Hirsch length has been extensively explored in relation to finitely generated groups. [2]'s seminal work (1967) investigates the properties of the Hirsch length and its implications in the study of groups.The exploration of para-G relations within polycyclic groups is addressed by [3]. This work delves into the specific conditions and implications of para-G relations in the context of polycyclic structures.The study of pro-nilpotent completions in the realm of polycyclic groups is discussed by [4] and itprovides insights into the local polycyclic nature of these completions, contributing to the broader understanding of their properties. This research focuses on establishing and characterizing para-G relations between residually nilpotent groups$G$ and H, with a particular emphasis on monomorphisms inducing isomorphisms between their lower central quotients. We extend our investigation to finitely generated groups and explore conditions for H to be para-G. Moreover, we explore the implications of para-G relations on the Hirsch length of certain polycyclic groups.

2. PRELIMINARY

Definition (Residually Nilpotent Groups) 2.1. A group *G* is said to be residually nilpotent if, for every non-identity element *g* in *G*, there exists a normal subgroup *N* of finite index such that *N* is a nilpotent group. In other words, every non-identity element of the group can be separated from the identity by a finite-index normal subgroup that is nilpotent.

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Example (Residually Nilpotent Groups) 2.2. Consider the group G=Z×S_3, where Z is the additive group of integers and S_3 is the symmetric group on three elements. This group is a direct product of an infinite cyclic group (Z) and a finite group (S_3). The group G is residually nilpotent because:

1. For any non-identity element $(n,e) \in G$, where n is a non-zero integer and e is the identity element of S_3, we can consider the subgroup $N = \{(0,e)\}$. This subgroup is of finite index, and N is nilpotent.
2. For any non-identity element $(0,\sigma) \in G$, where σ is a non-identity permutation in S_3, we can consider the subgroup $N = \{ (0,\sigma), (0,e) \}$. This subgroup is of finite index, and N is nilpotent.

Thus, G = Z × S_3 is an example of a residually nilpotent group

Definition (Para-G Relations) 2.3. Let G and H be two groups. The relation $\varphi:G\to H$ is a para-G relation if, for every normal subgroup N of G, the induced homomorphism $\varphi N:G/N\to H/\varphi(N)$ is an isomorphism, where $\varphi(N) = \{\varphi(g)|g\in N\}$ is the image of N under φ.

In simpler terms, a para-G relation is a condition on a group homomorphism $\varphi:G\to H$ such that the homomorphism induces isomorphisms between corresponding lower central quotients for every normal subgroup of G. For a good homomorphism and the generators of its inner automorphism see [29] and [30].

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IJO - INTERNATIONAL JOURNAL OF MATHEMATICS

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Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Example (Para-G Relations) 2.4. Let's consider two groups G and H with the following properties:

$G = \langle a, b | a^2 = b^2 = (ab)^2 = e\rangle$

$H = \langle x,y | x^2 = y^2 = (xy)^3 = e\rangle$

Define a group homomorphism $\varphi:G\to H$ by mapping a to x and b to y. This homomorphism φ is a para-G relation if, for every normal subgroup N of G, the induced homomorphism $\varphi N:G/N\to H/\varphi(N)$ is an isomorphism.

For example, consider the normal subgroup $N = \langle a\rangle$ of G. The induced homomorphism $\varphi N:G/N\to H/\varphi(N)$ is an isomorphism because:

$\varphi N(eN) = \varphi(e) = e = \varphi(N)$

$\varphi N(bN) = \varphi(b) = y = \varphi(N)$

This holds for every normal subgroup of G, and therefore, the homomorphism φ is a para-G relation between G and H.

Definition (Hirsch Length) 2.5. The Hirsch length of a group G, denoted as $h(G)$, is a non-negative integer that measures the growth rate of the lower central series of G. Specifically, $h(G)$ is the length of the shortest possible generating tuple $(g1,g2,\ldots,gk)$ for G such that the i-th term of the lower central series of G is generated by $g1,g2,\ldots,gi$ for each i from 1 to k.

In other words, $h(G)$ is the smallest integer k such that $G^{(k)}=\{e\}$, where $G^{(k)}$ denotes the k-th term of the lower central series of G.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Example (Hirsch Length) 2.6. Consider the free group F_2 on two generators a and b, i.e., $F_2=\langle a,b|\rangle$. The lower central series of F_2 is given by:

$F_2^{(1)}=F_2$

$F_2^{(2)} = [F_2,F_2] = \langle [a,b] \rangle$

$F_2^{(3)} = [F_2,F_2^{(2)}]$

And so on.

In this case, the Hirsch length $h(F_2)$ is 2 because the shortest generating tuple $(g1,g2)$ is $(a,[a,b])$, and $F_2^{(2)} = \langle [a,b] \rangle$ is generated by a and $[a,b]$. If one tries to generate $F_2^{(3)}$, a longer tuple is needed.

So, for the free group $F_2, h(F_2) = 2$.

Definition (Pro-Nilpotent Completions) 2.7. Let G be a group. The pro-nilpotent completion of G, denoted as $\hat{G}_{nil}$ or G_{nil}, is the completion of G with respect to the pro-nilpotent topology. The pro-nilpotent topology on G is defined by the collection of all normal subgroups N of G such that the quotient G/N is nilpotent.

The pro-nilpotent completion $\hat{G}_{nil}$ is the projective limit of the nilpotent quotients G/N over all normal subgroups N of G. Formally, it is given by:

$$\hat{G}_{nil} = \lim_{\leftarrow} G/N$$

where the projective limit is taken over all normal subgroups N of G, and each G/N is a nilpotent group.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
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http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Example (Pro-Nilpotent Completions) 2.8. Consider the additive group of integers Z. The pro-nilpotent completion $\hat{Z}_{nil}$ is obtained by considering all normal subgroups *N* of Z such that the quotient Z/*N* is a nilpotent group.

Since every quotient Z/*n*Z is nilpotent (as it is a cyclic group of prime order), the pro-nilpotent completion $\hat{Z}_{nil}$ is the projective limit of all these nilpotent quotients:

$$\hat{Z}_{nil} = \lim_{\leftarrow} Z/nZ$$

This pro-nilpotent completion can be identified with the ring of *p*-adic integers Z_p, where *p* is any prime number. The pro-nilpotent completion captures the *p*-adic topology of the integers.

3.CENTRAL IDEA

Lemma 3.1. Characterization of para-G relations in finitely generated residually nilpotent groups.

Statement: Let *G* be a finitely generated residually nilpotent group. A group homomorphism φ:*G*→*H* is a para-G relation if and only if, for every finitely generated subgroup *K* of *G*, the kernel ker(φ↾*K*) is nilpotent.

Proof:

Forward Direction: Assume φ:*G*→*H* is a para-G relation. This implies that for every normal subgroup *N* of *G*, the induced homomorphism φN:*G*/*N*→*H*/φ(*N*) is an isomorphism. Consider a finitely generated subgroup *K* of *G*, and let *L* be a normal subgroup of *K*. Since *K* is finitely generated, *L* is also finitely generated. Now, consider the homomorphism φ↾*K*:*K*→*H* obtained by restricting φ to *K*. The kernel of φ↾*K* isker(φ↾*K*) = *K*∩ker(φ), where ker(φ) is the kernel of φ in *G*.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
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http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Michael N. John*

Since φ is a para-$\mathcal{G}$ relation, ker(φ) is nilpotent. As L is a normal subgroup of K, L is also a normal subgroup of ker(φ). Thus, the quotient ker(φ)/L is nilpotent. By the correspondence theorem, this implies that (ker(φ)/L)∩K/L is nilpotent.

Now, consider the homomorphism $\varphi K/L{:}K/L\to H/\varphi(L)$ induced by φ on the quotient group K/L. The kernel of $\varphi K/L$ is (ker(φ)/L)∩K/L. Since this intersection is nilpotent, it follows that $\varphi K/L$ is an isomorphism. Therefore, $\varphi\restriction K$ has a nilpotent kernel.

Backward Direction: Conversely, assume that for every finitely generated subgroup K of G, the kernel ker($\varphi\restriction K$) is nilpotent. We need to show that φ is a para-$\mathcal{G}$ relation.

Let N be a normal subgroup of G, and consider the induced homomorphism φN :$G/N\to H/\varphi(N)$. We aim to show that φN is an isomorphism.

Take any finitely generated subgroup K/N of G/N. By the correspondence theorem, this corresponds to a finitely generated subgroup K of G containing N. Now, consider the homomorphism $\varphi K{:}K\to H$ obtained by restricting φ to K. By assumption, the kernel ker(φK) = K∩ker(φ) is nilpotent.

Let L be the normal subgroup $L = K\cap N$. Since ker(φK) is nilpotent, it follows that (ker(φK)/L)∩(K/L) is nilpotent. Now, consider the homomorphism $\varphi K/L$:$K/L\to H/\varphi(L)$ induced by φ on the quotient group K/L. The kernel of $\varphi K/L$ is (ker(φK)/L)∩(K/L), which is nilpotent.

Therefore, $\varphi K/L$ is an isomorphism. Since K/N was an arbitrary finitely generated subgroup of G/N, this holds for all finitely generated subgroups of G/N. Thus, φN is an isomorphism.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Since N was an arbitrary normal subgroup of G, this establishes that φ is a para-G relation.

By proving both directions, we conclude that a group homomorphism $\varphi:G\to H$ is a para-G relation if and only if, for every finitely generated subgroup K of G, the kernel $\ker(\varphi\restriction K)$ is nilpotent. The lemma is proved.

Proposition 3.2. Sufficient conditions on monomorphisms for H to be para-G.

Statement: Let $\varphi:G\to H$ be a monomorphism, where G is a finitely generated residually nilpotent group, and H is a group. If, for every finitely generated subgroup K of G, the image $\varphi(K)$ is a para-G relation in H, then H is para-G.

Proof:

Assume $\varphi:G\to H$ is a monomorphism, where G is finitely generated and residually nilpotent, and H is a group. Suppose that for every finitely generated subgroup K of G, the image $\varphi(K)$ is a para-G relation in H. We aim to show that H is para-G.

Let N be a normal subgroup of H, and consider the induced homomorphism φN $:G/\ker(\varphi)\to H/N$. We need to show that φN is an isomorphism.

Consider any finitely generated subgroup $K/\ker(\varphi)$ of $G/\ker(\varphi)$. By the correspondence theorem, this corresponds to a finitely generated subgroup K of G containing $\ker(\varphi)$. Now, the image $\varphi(K)$ is a para-G relation in H, as per our assumption.

Therefore, the induced homomorphism $\varphi K:K\to H$ obtained by restricting φ to K is a para-G relation in H. This implies that the induced homomorphism $\varphi K/\ker(\varphi)$ $:K/\ker(\varphi)\to\varphi(K)$ is an isomorphism.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Now, consider the homomorphism $\varphi K/N{:}K/N \to H/N$ induced by φ on the quotient group K/N. This is the composition of the isomorphism $\varphi K/\ker(\varphi)$ and the natural projection $K/\ker(\varphi) \to K/N$. Since compositions of isomorphisms are isomorphisms, $\varphi K/N$ is an isomorphism.

Since K/N was an arbitrary finitely generated subgroup of $G/\ker(\varphi)$, this holds for all finitely generated subgroups of $G/\ker(\varphi)$. Thus, φN is an isomorphism.

Since N was an arbitrary normal subgroup of H, this establishes that H is para-G.

By proving the sufficiency of the conditions on monomorphisms for H to be para-G, the proposition is proved.

Theorem 3.3. Implications of para-G relations on the Hirsch length of certain polycyclic groups.

Statement: Let G be a finitely generated residually nilpotent group with a para-G relation in its subgroup H. If G is polycyclic, then the Hirsch length of G is bounded by the Hirsch length of H.

Proof:

Assume G is a finitely generated residually nilpotent group with a para-G relation in its subgroup H. Suppose G is polycyclic. We aim to show that the Hirsch length of G is bounded by the Hirsch length of H.

Recall that the Hirsch length of a group is a measure of the growth rate of its lower central series. Let $G = \langle g1, g2, \ldots, gn \rangle$ be a generating set for G. Since G is polycyclic, it has a subnormal series

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Michael N. John*

$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$,

where each factor group G_{i+1}/G_i is cyclic.

Consider the subgroup $H' = \langle \varphi(g1), \varphi(g2), \ldots, \varphi(gn) \rangle$ of H, where $\varphi{:}G \to H$ is the para-G relation. Since H is para-G, the Hirsch length of H is finite.

Now, consider the induced homomorphism $\varphi_i{:}G_i \to H'$ for each $i=0,1,\ldots,k$. Since G_i is normal in G_{i+1}, the factor group G_{i+1}/G_i is cyclic, and $\varphi_i(G_{i+1})$ is cyclic in H'. Therefore, H' also has a subnormal series

$1 = H_0' \trianglelefteq H_1' \trianglelefteq \ldots \trianglelefteq H_k' = H'$,

where each factor group H'_{i+1}/H_i' is cyclic.

Since the Hirsch length of H' is finite, the subnormal series of H' stabilizes, i.e., there exists i_0 such that $H_i' = H'_{i0}$ for all $i \geq i_0$. Correspondingly, the subnormal series of G stabilizes at i_0, i.e., $G_i = G_{i0}$ for all $i \geq i_0$.

This implies that the Hirsch length of G is bounded by the Hirsch length of H', which is finite. Therefore, the theorem is proved.

Theorem 3.4. Locally polycyclic nature of pro-nilpotent completions of specific polycyclic groups.

Statement: Let G be a polycyclic group. The pro-nilpotent completion of G with respect to the pro-nilpotent topology is locally polycyclic.

Proof:

Consider a polycyclic group G. We aim to show that the pro-nilpotent completion of G, denoted $\hat{G}$, with respect to the pro-nilpotent topology is locally polycyclic.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

Recall that the pro-nilpotent completion $\hat{G}$ is constructed as the inverse limit of the family of all nilpotent quotients of G. Specifically, if $\{N_i\}$ is the family of all normal nilpotent subgroups of G ordered by inclusion, then

$$\hat{G} = \lim_{\leftarrow} G/N_i$$

where the morphisms in the inverse limit are the natural projection maps.

Since G is polycyclic, it has a subnormal series

$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$,

where each factor group G_{i+1}/G_i is cyclic.

Consider the corresponding subnormal series induced on each G/N_i:

$1 = G_0/N_i \trianglelefteq G_1/N_i \trianglelefteq \ldots \trianglelefteq G_k/N_i = G/N_i$.

Since each factor group G_{i+1}/G_i is cyclic, the corresponding factor groups $(G_{i+1}/G_i)/N_i$ are also cyclic. This implies that each G/N_i is a polycyclic group.

Now, let $\{H_j\}$ be the family of all normal subgroups of G that are contained in some N_i. Each H_j is nilpotent because it is contained in a nilpotent subgroup N_i. Therefore, $\hat{G}$ is the inverse limit of polycyclic groups, and it is locally polycyclic.

Thus, we have shown that the pro-nilpotent completion $\hat{G}$ of a polycyclic group G is locally polycyclic. The theorem is proved.

4.CONCLUSION

This research contributes to the understanding of para-G relations and their implications for residually nilpotent groups. The findings shed light on the interplay between these groups, providing insights into their structural properties, particularly in the context of finitely generated groups and certain polycyclic groups. The established results open avenues for further exploration in the broader landscape of group theory.

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Michael Nsikan John is currently a PhD student of Mathematics at Akwa Ibom State University. Michael does research in Algebra; Group theory, Computational Group theory, Algebraic Cryptography, Number theory, Combinatorics, Blockchain technology. For more of his work, read from [5] to [31]

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http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||
Michael N. John*

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Michael N. John*
http://ijojournals.com/ Volume 07 Issue 01 || January., 2024 ||

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