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ADVANCEMENTS IN LINEAR MULTI-STEP METHOD FOR SOLVING THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

This work addresses the development of four step linear multi-step methods for the solution of third order ordinary differential equations. The approach requires the construction of a truncation error term and expanding it in Taylor's series. The resulting FOUR step method are analysed to show that it is consistent, zero stable and hence convergent with good interval of absolute stability.Thus the new method satisfies the minimum condition for a linear multi-step method to be acceptable. The technique of derivation employed in this work is easier and more adaptable than those of collocation

Keywords: Four-Step Method, Third-Order Ordinary Differential Equations, Truncation Error, Taylor's Series, Consistency, Zero Stability, Convergence, Absolute Stability, Numerical Analysis.

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1. INTRODUCTION

Ordinary differential equations (ODEs) are fundamental in modeling dynamic systems across various scientific disciplines. [1]'s paper discusses the construction of linear multistep methods, providing insights into the techniques used for their development and analysis.Burden and Faires' [2] textbook is a comprehensive resource for numerical analysis. Chapter discussions on multistep methods offer foundational knowledge in the field.Butcher's book [3] is a classic in the field, providing a deep understanding of various numerical methods for ordinary differential equations, including multistep methods. [4]'s book covers the computational aspects of ordinary differential equations, providing valuable insights into the development and analysis of numerical methods. The first volume of [5] series delves into the numerical solution of nonstiff ordinary differential equations, offering relevant information for the development of multistep methods.Lambert's work [6] is a foundational resource on computational methods for ordinary differential equations, providing a historical context for the development of numerical techniques.Shampine and Gordon's book [7] is a classic in the field, providing practical insights into the numerical solution of ordinary differential equations, including the development of multistep methods. This paper presents a novel contribution to the field by introducing a four-step linear multi-step method for solving third-order ODEs. The methodology involves the construction of a truncation error term, which is then expanded using Taylor's series.

The resulting four-step method undergoes a thorough analysis to establish its key properties. We demonstrate its consistency, ensuring an accurate representation of the underlying differential equation. Moreover, we prove its zero stability, indicating reliable behavior, and establish its convergence with a substantial interval of absolute stability. This work represents a significant

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advancement in numerical methods for ODEs, providing an efficient and acceptable solution to the complex challenges posed by third-order equations.

2. METHODOLOGY

This section describes the development of a foursteplinear multi-step method for the solution of initial value problems of ordinary differential equation.

MethodsOf Derivation Of The New Linear Multi-Step 2.1

i. The L. H. S is expanded by Taylor's series about h
ii. The R. H. S i.e. y_{n+i} is expanded in Taylor's series about h.
iii. Replace the function f_{n+j} with y'''_{n+j} and then expand in Taylor's about h
iv. Put the system in matrix form
v. Determine the values of $\alpha's$ and $\beta's$
vi. Finally form the new linear multi-step method
vii. Test the linear multi-step method for convergence.

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From

$$y_{n+k} = \sum_{i=0}^{3} \alpha_i \, y_{n+i} + h^3 \sum_{j=o}^{4} \beta_{j\, f_{n+j}} \qquad (2.1)$$

Implies

$$y_{n+k} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + h^3[\beta_0 y_n''' + \beta_1 y_{n+1}''' + \beta_2 y_{n+2}''' + \beta_3 y_{n+3}''' + \beta_4 y_{n+4}'''](2.2)$$

Expanding the L.H.S.by Taylor's series, about h and when k = 4, we obtain;

$$y_{n+4} = \frac{(4h)^0}{0!} y_n + \frac{(4h)^1}{1!} y'_n + \frac{(4h)^2}{2!} y_n'' + \frac{(4h)^3}{3!} y_n''' + \frac{(4h)^4}{4!} y_n^{iv} + \frac{(4h)^5}{5!} y_n^{v} + \frac{(4h)^6}{6!} y_n^{vi} + \frac{(4h)^7}{7!} y_n^{vii} + \frac{(4h)^8}{8!} y_n^{viii} + \cdots$$

$$y_{n+4} = 4^0 h^o y_n + 4^1 h^1 y_n' + \frac{4^2 h^2}{2} y_n'' + \frac{4^3 h^3}{6} y_n''' + \frac{4^4 h^4}{24} y_n^{iv} + \frac{4^5 h^5}{120} y_n^{v} + \frac{4^6 h^6}{720} y_n^{vi} + \frac{4^7 h^7}{5040} y_n^{vii} + \frac{4^8 h^8}{40320} y_n^{viii} + \frac{4^9 h^9 y_n^{ix}}{362880} + \cdots$$

$$y_{n+4} = h^0 y_n + 4h y_n' + \frac{16h^2}{2} y_n'' + \frac{64h^3}{6} y_n''' + \frac{256h^4}{24} y_n^{iv} + \frac{1024h^5}{120} y_n^{v} + \frac{4096h^6}{720} y_n^{vi} + \frac{16384h^7}{5040} y_n^{vii} + \frac{65536h^8}{40320} y_n^{viii} + \frac{262144h^9 y_n^{ix}}{362880} + \qquad (2.3)$$

Expanding the coefficients of $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 and\ \beta_4$by Taylor's series about h as in equation(2.1), we obtain;

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$$\alpha_0 y_n h^0 + \alpha_1\left(y_n h^0 + y_n' h + \frac{y_n''}{2!}h^2 + \frac{y_n'''}{3!}h^3 + \frac{y_n^{iv}}{4!}h^4 + \frac{y_n^{v}}{5!}h^5 + \frac{y_n^{vi}}{6!}h^6 + \frac{y_n^{vii}}{7!}h^7 + \frac{y_n^{viii}}{8!}h^8 + \frac{y_n^{ix}}{9!!}h^9 + \frac{y_n^{x}}{10!}h^{10} \ldots\right)$$

$$+ \alpha_2\left(y_0 h^0 + 2y_n' h + \frac{4y_n''}{2!}h^2 + \frac{8y_n'''}{3!}h^3 + \frac{16y_n^{iv}}{4!}h^4 + \frac{32y_n^{v}}{5!}h^5 + \frac{64y_n^{vi}}{6!}h^6 + \frac{128y_n^{vii}}{7!}h^7 + \frac{256y_n^{viii}h^8}{8!} + \frac{512y_n^{ix}h^9}{9!}\right.$$

$$\left. + \frac{1024y_n^{x}}{10!}h^{10} \ldots\right)$$

$$+ \alpha_3\left(\frac{(3h)^0 y_n}{0!}h^0 + \frac{(3h)^1 y_n'}{1!}h^1 + \frac{(3h)^2 y_n''}{2!}h^2 + \frac{(3h)^3 y_n'''}{3!}h^3 + \frac{(3h)^4 y_n^{iv}}{4!}h^4 + \frac{(3h)^5 y_n^{v}}{5!}h^5 + \frac{(3h)^6 y_n^{vi}}{6!}h^6 + \frac{(3h)^7 y_n^{vii}}{7!}h^7\right.$$

$$\left. + \frac{(3h)^8 y_n^{viii}}{8!}h^8 + \frac{(3h)^9 y_n^{ix}}{9!}h^9 + \frac{(3h)^{10} y_n^{x}}{10!}h^{10} + \cdots\right)$$

$$+ h^3\left[\beta_0 y_n''' h^0 + \beta_1\left\{y_n''' h^0 + y_n^{iv} h + \frac{y_n^{v}}{2!}h^2 + \frac{y_n^{vi}}{3!}h^3 + \frac{y_n^{vii}}{4!}h^4 + \frac{y_n^{viii}}{5!}h^5 + \frac{y_n^{ix}}{6!}h^6 + \frac{y_n^{x}}{7!}h^7 + \cdots\right\}\right.$$

$$+ \beta_2\left\{y_n''' h^0 + 2y_n^{iv} h + \frac{4y_n^{v}}{2!}h^2 + \frac{8y_n^{vi}}{3!}h^3 + \frac{16y_n^{vii}}{4!}h^4 + \frac{128y_n^{x}h^7}{7!} + \cdots\right\}$$

$$+ \beta_3\left\{y_n''' h^0 + 3y_n^{iv} h + \frac{9y_n^{v}}{2!}h^2 + \frac{27y_n^{vi}}{3!}h^3 + \frac{81y_n^{vii}}{4!}h^4 + \frac{243y_n^{viii}}{5!}h^5 + \frac{729y_n^{ix}}{6!}h^6 + \frac{2187y_n^{x}}{7!}h^7 + \cdots\right\}$$

$$\left. + \beta_4\left\{\frac{(4h)^0 y_n'''}{0!} + \frac{(4h)^1 y_n'^{v}}{1!} + \frac{(4h)^2 y_n^{v}}{2!} + \frac{(4h)^3 y_n^{vi}}{3!} + \frac{(4h)^4 y_n^{vii}}{4!} + \frac{(4h)^5 y_n^{viii}}{5!} + \frac{(4h)^6 y_n^{ix}}{6!} + \frac{(4h)^7 y_n^{x}}{7!} + \cdots\right\}\right]$$

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By clearing the bracket, we have;

$$\alpha_0 y_n h^0 + \alpha_1 y_n h^0 + \alpha_1 y_n' h + \alpha_1 \frac{y_n''}{2} h^2 + \alpha_1 \frac{y_n'''}{6} h^3 + \alpha_1 \frac{y_n^{iv}}{24} h^4 + \alpha_1 \frac{y_n^{v}}{120} h^5 + \alpha_1 \frac{y_n^{vi}}{720} h^6$$
$$+ \alpha_1 \frac{y_n^{vii}}{5040} h^7 + \cdots + \alpha_2 y_n h^0 + 2\alpha_2 y_n' h + 2\alpha_2 y_n'' h^2 + \frac{8\alpha_2 y_n'''}{6} h^3 + \frac{16\alpha_2 y_n^{iv}}{24} h^4$$
$$+ \frac{32\alpha_2 y_n^{v}}{120} h^5 + \frac{64\alpha_2 y_n^{vi}}{720} h^6 + \frac{128\alpha_2 y_n^{vii}}{5040} h^7 + \alpha_3 h^0 y_n + 3\alpha_3 h y_n' + \frac{9}{2}\alpha_3 h^2 y_n''$$
$$+ \frac{27}{6}\alpha_3 h^3 y_n''' + \frac{81}{24}\alpha_3 h^4 y_n^{iv} + \frac{243}{120}\alpha_3 h^5 y_n^{v} + \frac{729}{720}\alpha_3 h^6 y_n^{vi} + \frac{2187}{5040}\alpha_3 h^7 y_n^{vii}$$
$$+ \frac{6561}{40320}\alpha_3 h^8 y_n^{viii} + \frac{19683}{362880}\alpha_3 h^9 y_n^{ix} + \frac{59049}{3628800}\alpha_3 h^{10} y_n^{x} \ldots + \beta_0 y_n''' h^3$$
$$+ \beta_1 y_n^{iv} h^3 + \frac{\beta_1 y_n^{v}}{2} h^5 + \frac{\beta_1 y_n^{vi}}{6} h^6 + \frac{\beta_1 y_n^{vii}}{24} h^7 + \frac{\beta_1 y_n^{viii}}{120} h^8 + \frac{\beta_1 y_n^{ix}}{720} h^9$$
$$+ \frac{\beta_1 y_n^{x}}{5040} h^{10} + \frac{h^{11}\beta_1 y_n^{xi}}{40320} + \frac{h^{12}\beta_1 y_n^{xii}}{362880} + \frac{h^{13}\beta_1 y_n^{xiii}}{3628800} + \cdots + \beta_2 y_n^{iii} h^3 + 2\beta_2 y_n^{iv} h^4$$
$$+ 2\beta_2 y_n^{v} h^5 + \frac{8 y_n^{vi}}{6} h^6 + \frac{16\beta_2 y_n^{vii}}{24} h^7 + \frac{32\beta_2 y_n^{viii}}{120} h^8 + \frac{64\beta_2 y_n^{ix}}{720} h^9$$
$$+ \frac{128\beta_2 y_n^{x}}{5040} h^{10} + \cdots + \beta_3 y_n''' h^3 + 3\beta_3 y_n^{iv} h^4 + \frac{9\beta_3 {y_n}^{v}}{2} h^5 + \frac{27\beta_3 y_n^{vi}}{6} h^6$$
$$+ \frac{81\beta_3 y_n^{vii}}{24} h^7 + \frac{243\beta_3 y_n^{viii}}{120} h^8 + \frac{729 y_n^{ix}\beta_3}{270} h^9 + \frac{2187\beta_3 y_n^{x}}{5040} h^{10} + \cdots + h^3 \beta_4 y_n$$
$$+ 4h^4\beta_4 y_n' + \frac{16}{2} h^5 \beta_4 y_n'' + \frac{64}{6} h^6 \beta_4 y_n''' + \frac{256 h^7 \beta_4 y_n^{iv}}{24} + \frac{1024 h^8 \beta_4 y_n^{v}}{120}$$
$$+ \frac{4096 h^9 \beta_4 y_n^{vi}}{720} + \qquad (2.4)$$

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Equating equation (2.3) and equation (2.4), we have;

$$y_{n+4} = h^0 y_n + 4hy_n' + \frac{16h^2}{2}y_n'' + \frac{64h^3}{6}y_n''' + \frac{256h^4}{24}y_n^{iv} + \frac{1024h^5}{120}y_n^{v} + \frac{4096h^6}{720}y_n^{vi} + \frac{16384h^7}{5040}y_n^{vii} + \frac{65536h^8}{40320}y_n^{viii} + \frac{262144h^9 y_n^{ix}}{362880} + \cdots$$

$$= \alpha_0 y_n h^0 + \alpha_1 y_n h^0 + \alpha_1 y_n' h + \alpha_1 \frac{y_n''}{2}h^2 + \alpha_1 \frac{y_n'''}{6}h^3 + \alpha_1 \frac{y_n^{iv}}{24}h^4 + \alpha_1 \frac{y_n^{v}}{120}h^5 + \alpha_1 \frac{y_n^{vi}}{720}h^6 + \alpha_1 \frac{y_n^{vii}}{5040}h^7 + \cdots + \alpha_2 y_n h^0$$

$$+ 2\alpha_2 y_n' h + 2\alpha_2 y_n'' h^2 + \frac{8\alpha_2 y_n'''}{6}h^3 + \frac{16\alpha_2 y_n^{iv}}{24}h^4 + \frac{32\alpha_2 y_n^{v}}{120}h^5 + \frac{64\alpha_2 y_n^{vi}}{720}h^6 + \frac{128\alpha_2 y_n^{vii}}{5040}h^7 + \cdots + \alpha_3 h^0 y_n + 3\alpha_3 h y_n'$$

$$+ \frac{9}{2}\alpha_3 h^2 y_n'' + \frac{27}{6}\alpha_3 h^3 y_n''' + \frac{81}{24}\alpha_3 h^4 y_n^{iv} + \frac{243}{120}\alpha_3 h^5 y_n^{v} + \frac{729}{720}\alpha_3 h^6 y_n^{vi} + \frac{2187}{5040}\alpha_3 h^7 y_n^{vii} + \frac{6561}{40320}\alpha_3 h^8 y_n^{viii}$$

$$+ \frac{19683}{362880}\alpha_3 h^9 y_n^{ix} + \frac{59049}{3628800}\alpha_3 h^{10} y_n^{x} + \beta_0 y_n''' h^3 + \beta_1 y_n^{iii} h^3 + \beta_1 y_n^{iv} h^4 + \frac{\beta_1 y_n^{v}}{2}h^5 + \frac{\beta_1 y_n^{vi}}{6}h^6 + \frac{\beta_1 y_n^{vii}}{24}h^7$$

$$+ \frac{\beta_1 y_n^{viii}}{120}h^8 + \frac{\beta_1 y_n^{ix}}{720}h^9 + \frac{\beta_1 y_n^{x}}{5040}h^{10} + \cdots + \beta_2 y_n^{iii} h^3 + 2\beta_2 y_n^{iv} h^4 + 2\beta_2 y_n^{v} h^5 + \frac{8y_n^{vi}}{6}h^6 + \frac{16\beta_2 y_n^{vii}}{24}h^7 + \frac{32\beta_2 y_n^{viii}}{120}h^8$$

$$+ \frac{64\beta_2 y_n^{ix}}{720}h^9 + \frac{128\beta_2 y_n^{x}}{5040}h^{10} + \cdots + \beta_3 y_n''' h^3 + 3\beta_3 y_n^{iv} h^4 + \frac{9\beta_{3y}{}_n^{v}}{2}h^5 + \frac{27\beta_3 y_n^{vi}}{6}h^6 + \frac{81\beta_3 y_n^{vii}}{24}h^7 + \frac{243\beta_3 y_n^{viii}}{120}h^8$$

$$+ \frac{729 y_n^{ix}}{270}h^9 + \frac{2187\beta_3 y_n^{x}}{5040}h^{10} + \cdots + h^3\beta_4 y_n + 4h^4\beta_4 y_n' + \frac{16}{2}h^5\beta_4 y_n'' + \frac{64}{6}h^6\beta_4 y_n''' + \frac{256h^7\beta_4 y_n^{iv}}{24} + \frac{1024h^8\beta_4 y_n^{v}}{120}$$

$$+ \frac{4096h^9\beta_4 y_n^{vi}}{720} + \cdots$$

By comparing the coefficient in powers of h, we obtain;

i. $h^0 \Rightarrow \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$

ii. $h^1 \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 4$

iii. $h^2 \Rightarrow \frac{\alpha_1}{2} + 2\alpha_2 + \frac{9}{2}\alpha_3 = \frac{16}{2}$

iv. $h^3 \Rightarrow \frac{\alpha_1}{6} + \alpha_2\frac{8}{6} + \frac{27}{6}\alpha_3 + \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{64}{6}$

v. $h^4 \Rightarrow \frac{\alpha_1}{24} + \alpha_2\frac{16}{24} + \frac{81}{24}\alpha_3 + \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 = \frac{256}{24}$

vi. $h^5 \Rightarrow \frac{\alpha_1}{120} + \alpha_2\frac{32}{120} + \frac{243}{120}\alpha_3 + \frac{\beta_1}{2} + 2\beta_2 + \beta_3\frac{9}{2} + \frac{16}{2}\beta_4 = \frac{1024}{120}$

vii. $h^6 \Rightarrow \frac{\alpha_1}{720} + \alpha_2\frac{64}{720} + \frac{729}{720}\alpha_3 + \frac{\beta_1}{6} + \beta_2\frac{8}{6} + \beta_3\frac{27}{6} + \frac{64}{6}\beta_4 = \frac{4096}{720}$

viii. $h^7 \Rightarrow \frac{\alpha_1}{5040} + \frac{128\alpha_2}{5040} + \frac{2187\alpha_3}{5040} + \frac{\beta_1}{24} + \frac{16\beta_2}{24} + \frac{81\beta_3}{24} + \frac{256\beta_4}{24} = \frac{16384}{5040}$

ix. $h^8 \Rightarrow \frac{\alpha_1}{40320} + \frac{256\alpha_2}{40320} + \frac{6561\alpha_3}{40320} + \frac{\beta_1}{120} + \frac{32\beta_2}{120} + \frac{243\beta_3}{120} + \frac{1024\beta_4}{120} = \frac{65536}{40320}$

$(i - ix)(2.5)$

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Transforming the above into matrix form, yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{4}{2} & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{8}{2} & \frac{27}{6} & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{24} & \frac{16}{24} & \frac{81}{24} & 0 & 1 & 2 & 3 & 4 \\ 0 & \frac{1}{120} & \frac{32}{120} & \frac{243}{120} & 0 & \frac{1}{2} & \frac{4}{2} & \frac{9}{2} & \frac{16}{2} \\ 0 & \frac{1}{170} & \frac{64}{720} & \frac{729}{720} & 0 & \frac{1}{6} & \frac{8}{6} & \frac{27}{6} & \frac{64}{6} \\ 0 & \frac{1}{5040} & \frac{128}{5040} & \frac{2187}{5040} & 0 & \frac{1}{24} & \frac{16}{24} & \frac{81}{24} & \frac{256}{24} \\ 0 & \frac{1}{40320} & \frac{256}{40320} & \frac{6561}{40320} & 0 & \frac{1}{120} & \frac{32}{120} & \frac{243}{120} & \frac{1024}{120} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \frac{16}{2} \\ \frac{64}{6} \\ \frac{256}{24} \\ \frac{1024}{120} \\ \frac{4096}{720} \\ \frac{16384}{5040} \\ \frac{65536}{40320} \end{bmatrix} \quad \dots (2.6)$$

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3. ANALYSIS OF BASIC PROPERTIES OF THE FOUR STEP METHOD

This section seek to establish the basic properties of the linear multistep method as stated in chapter one.

Properties Of The Four Step Method 3.1

Order and Error Constant

From equation (2.5), we obtain the follow:

$D_0 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$

Substituting 1, -2, 0,and 2 for $\alpha_0, \alpha_1, \alpha_2$ *and* α_3 above we have

$D_0 = 1 - 1 - (-2) - 0 - 2$

$D_0 = 1 - 1 + 2 - 2$

$D_0 = 0.$

$D_1 = 4 - \alpha_1 - 2\alpha_2 - 3\alpha_3$

Whichimplies that

$D_1 = 4 - (-2) - 2(0) - 3(2)$

$D_1 = 0$

$D_2 = \frac{16}{2} - \frac{\alpha_1}{2} - 2\alpha_2 - \frac{9}{2}\alpha_3$

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Which implies that

$$D_2 = \frac{16}{2} - \frac{1}{2}(-2) - 2(0) - \frac{9}{2}(2)$$

Taking the L.C.M, we have

$$D_2 = \frac{16 + 2 - 18}{2}$$

Therefore

$$D_2 = 0.$$

$$D_3 = \frac{64}{6} - \frac{1}{6}\alpha_1 - \frac{8}{6}\alpha_2 - \frac{27}{6}\alpha_3 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4$$

By Substitution and for $\frac{1}{120}, \frac{7}{15}, \frac{21}{20}, \frac{7}{15},, \frac{1}{120} = \beta_0, \beta_1, \beta_2 \beta_3, \beta_4$ we have

$$D_3 = \frac{1280 + 40 - 1080 - 1 - 56 - 126 - 56 - 1}{120}$$

Therefore

$$D_3 = \frac{-7}{15}$$

$$D_4 = \frac{256}{24} - \frac{\alpha_1}{24} - \alpha_2 \frac{16}{24} - \frac{81}{24}\alpha_3 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4$$

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Substituting thevalue of the coefficient matrix, we have

$$D_4 = \frac{1280 + 10 - 3105 - 56 - 252 - 168 - 4}{120}$$

Therefore

$$D_4 = \frac{-2295}{120}$$

$$D_5 = \frac{1024}{120} - \frac{\alpha_1}{120} - \alpha_2 \frac{32}{120} - \frac{243}{120}\alpha_3 - \frac{\beta_1}{2} - 2\beta_2 - \beta_3 \frac{9}{2} - \frac{16}{2}\beta_4$$

Substituting the values of the coefficientmatrix, we have

$$D_5 = \frac{2048 + 4 - 972 - 560 - 504 - 504 - 16}{120}$$

And therefore

$$D_5 = \frac{-42}{10}.$$

$$D_6 = \frac{4096}{720} - \frac{\alpha_1}{720} - \alpha_2 \frac{64}{720} - \frac{729}{720}\alpha_3 - \frac{\beta_1}{6} - \beta_2 \frac{8}{6} - \frac{27}{6} - \frac{64}{6}\beta_4$$

By substitution, we have

$$D_6 = \frac{4096}{720} + \frac{2}{720} - \frac{1458}{720} - \frac{7}{90} - \frac{168}{120} - \frac{189}{90} - \frac{64}{720}$$

Therefore we have that

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$D_6 = 0$.

$$D_7 = \frac{16384}{5040} - \frac{\alpha_1}{5040} - \frac{128\alpha_2}{5040} - \frac{2187\alpha_3}{5040} - \frac{\beta_1}{24} - \frac{16\beta_2}{24} - \frac{81\beta_3}{24} - \frac{256\beta_4}{24}$$

Similarly upon substitution, we obtain that

$D_7 = 0.$

Finally we have

$$D_8 = \frac{65536}{40320} - \frac{\alpha_1}{40320} - \frac{256\alpha_2}{40320} - \frac{6561\alpha_3}{40320} - \frac{\beta_1}{120} - \frac{32\beta_2}{120} - \frac{243\beta_3}{120} - \frac{1024\beta_4}{120}$$

Substituting the value of the coefficient matrix, we have

$D_8 = \frac{-1}{40320} = D_{p+1} \neq 0.$

Hence we can deduce that our four step method is of order p = 5 with error constant

$D_{p+1} = \frac{-1}{40320}.$

Stability and Consistency 3.2

Form

$$\sum_{i=0}^{k-1} \alpha_i y_{n+i} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}$$

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, for $k = 4$, if we substitute the coefficients 1,-2,0, 2,$\frac{1}{120}$,$\frac{7}{15}$,$\frac{21}{20}$,$\frac{7}{15}$,$\frac{1}{120}$ respectively, for

$\alpha_0, \alpha_1, \alpha_2, \alpha_{3,}\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$, we have

$$y_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 + h^3[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4]$$

Which implies, that

$$y_{n+4} - 2y_{n+3} + 2y_{n+1} - y_n = h^3\left[\frac{1}{120} f_{n+4} + \frac{7}{15} f_{n+3} + \frac{21}{20} f_{n+2} + \frac{7}{15} f_{n+1} + \frac{1}{120} f_n\right]$$

Taking the L.C.M of the terms inside the square bracket, we have

$$y_{n+4} - 2y_{n+3} + 2y_{n+1} - y_n = \frac{1}{120}[f_{n+4} + 56\, f_{n+3} + 126\, f_{n+2} + 56\, f_{n+1} + f_n] \quad \dots (3.1)$$

From equation(3.1), the first characteristic polynomial denoted by $\rho(r)$ is:

$$\rho(r) = r^4 - 2r^3 + 2r^1 - r^0 \quad \dots (3.2)$$

Whichimplies that

$$\rho(r) = r^4 - 2r^3 + 2r - 1 \quad (3.3)$$

and the second characteristic polynomial denoted by $\delta(r)$ is:

$$\delta(r) = \frac{1}{120}[r^4 + 56r^3 + 126r^2 + 56r^1 + r^0](3.4)$$

where h^3 is ignored,

Which implies that

$$\delta(r) = \frac{1}{120}[r^4 + 56r^3 + 126r^2 + 56r + 1](3.5)$$

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Consistency has it that a linear multi-step method must satisfies the following properties:

(1) $\rho(1) = 0$ and

(2)$\rho'''(1) = 3!\,\delta(1)$.

We now test conditions (1) and (2)by using(3.3) as follows:

1. $\rho(1) = 1^4 - 2(1^3) + 2(1) - 1$

   Which implies that

   $\rho(1) = 1 - 2 + 2 - 1$

   And therefore

   $\rho(1) = 0$,

   This verifies condition 1.

   Taking the first derivative of (3.3), we have

   $$\rho'(r) = 4r^3 - 6r^2 + 2 \qquad (3.6)$$

   For r = 1 we have

   $\rho'(1) = 4.1^2 - 6(1^2) + 2$

Which implies that

   $\rho'(1) = 4 - 6 + 2$

   Therefore,

   $\rho'(1) = 0$

2. $\rho'''(1) = 3!\,\delta(1)$

   We need the second and third derivatives of (3.6),thus

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$\rho''(r) = 12r^2 - 12r$

Implying,

$\rho'''(r) = 24r - 12$, then

$\rho'''(1) = 24 - 12 = 12.$

and

$\delta(r) = \frac{1}{120}[r^4 + 56r^3 + 126r^2 + 56r + 1]$ 3!

Which implies that

$\delta(1) = \frac{1}{120}[1^4 + 561^3 + 1261^2 + 56 + 1]$ 3!

$$\delta(1) = \frac{1}{120}[1440]$$

Without loss of generality

$\delta(1) = 12$

$\rho'''(1) = \delta(1) = 12.$

This result verifies condition 2.

It has been established that our fourstep method satisfies the conditions1 and 2, hence it is convergent.

The first characteristic polynomial

$\rho(r) = r^4 - 2r^3 + 2r - 1,$

gives the possible values of r when $\rho(r)$ is zero to be $(r-1)^4 = 0.$

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Which implies that $r = 1,1,1$. Therefore, the four-step method satisfies the root condition, hence it is zero-stable.

Interval of absolute stability 3.3

We now seek to obtain the interval of absolute stability; this is done by applying the boundary locus method which is define as:

$h(r) = \frac{\rho(r)}{\delta(r)}$ (3.7)

where$\rho(r)$ is the first characteristic polynomial and $\delta(r)$ is the second characteristic polynomial.

From equation(3.3)and (3.5)

$\rho(r) = r^4 - 2r^3 + 2r - 1$ and $\delta(r) = \frac{1}{120}[r^4 + 56r^3 + 126r^2 + 56r + 1]$.

Then from (3.7) , We have

$$h(r) = \frac{120[r^4 - 2r^3 + 2r - 1]}{r^4 + 56r^3 + 126r^2 + 56r + 1} \quad (3.8)$$

byDeMoiveries's theorem

$z^n = e^{in\theta} = cosn\theta + isinn\theta$ where $cosn\theta$ is the real part of $e^{in\theta}$ and $sinn\theta$ is the imaginary part of$e^{in\theta}$,

but here we will replace z^n with r^n. So

$r^n = cosn\theta + isinn\theta$

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For n = 1 we have

$$r^1 = cos\theta + isin\theta$$

Which implies that

$$r = cos\theta + isin\theta \quad (3.9)$$

For n = 2, we have

$$r^2 = cos2\theta + isin2\theta \quad (3.10)$$

For n = 3, we have

$$r^3 = cos3\theta + isin3\theta \quad (3.11)$$

and

For n = 4 we obtain

$$r^4 = cos4\theta + isin4\theta \quad (3.12)$$

Substituting equation(3.9)– (3.12)into(3.8), we have

$$h(\theta) = \frac{120[cos4\theta + isin4\theta - 2(cos3\theta + isin3\theta) + 2(cos\theta + isin\theta) - 1]}{cos4\theta + isin4\theta + 56(cos3\theta + isin3\theta) + 126(cos2\theta + isin3\theta) + 56(cos\theta + isin\theta) + 1}$$

which implies that

$$h(\theta) = \frac{120[(cos4\theta - 2cos3\theta + 2cos2\theta - 1) + i(sin4\theta - 2sin3\theta + 2sin\theta)]}{[(cos4\theta + 56cos3\theta + 126cos2\theta + 56cos\theta + 1 + i(sin4\theta + 56sin3\theta + 126sin2\theta + 56sin\theta))]}$$

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By rationalization, we have

$$h(\theta)=\frac{120[(cos4\theta-2cos3\theta+2cos\theta-1)+i(sin4\theta-2sin3\theta+2sin\theta)]\;[(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)-i(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)]}{[(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta)+i(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)]\;[(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)-i(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)]}$$

which implies that

$$h(\theta)=\frac{\begin{gathered}120[(cos4\theta-2cos3\theta+2cos\theta-1)(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)]\\-120i[(cos4\theta-2cos3\theta+2cos\theta-1)(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)]\\+120i\begin{bmatrix}(sin4\theta-2sin3\theta+2sin\theta)(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)\\+120[(sin4\theta-2sin3\theta+2sin\theta)(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)]\end{bmatrix}\end{gathered}}{\begin{bmatrix}(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)\\+i(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)\\-i(cos4\theta+56cos3\theta+126cos2\theta+56cos\theta+1)(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)\\+(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)(sin4\theta+56sin3\theta+126sin2\theta+56sin\theta)\end{bmatrix}}$$

Opening the bracket of the numerator and that of the denominator, we have

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$$h(\theta)=\frac{\left[\begin{array}{c}120cos^2 4\theta+6720cos4\theta cos3\theta+15120cos4\theta cos2\theta+6720cos4\theta cos\theta+120cos4\theta-240cos4\theta cos3\theta\\ -13440cos^2 3\theta-30240cos3\theta cos2\theta-13440cos3\theta cos\theta-240cos3\theta+240cos\theta cos4\theta\\ +13440cos\theta cos3\theta+30240cos\theta cos2\theta+13440cos\theta cos^2\theta+240cos\theta-120cos4\theta-6720cos3\theta\\ -15120cos2\theta\\ -6720cos\theta-120-120icos4\theta sin4\theta-6720icos4\theta sin3\theta-15120icos4\theta sin2\theta-\\ 6720icos4\theta sin\theta+240icos3\theta sin4\theta\\ +13440icos3\theta sin3\theta+30240icos3\theta sin2\theta\\ +13440icos3\theta sin\theta\\ -240icos\theta sin4\theta-13440icos\theta sin3\theta\\ -13440icos\theta sin\theta+120isin4\theta\\ +120isin3\theta+15120isin2\theta+6720isin\theta 120isin4\theta cos4\theta\\ +6720isin4\theta cos3\theta\\ +15120icos2\theta sin4\theta+6720isin4\theta cos\theta\\ +120isin4\theta-240icos4\theta sin3\theta-13440i\,sin3\theta cos3\theta\\ -30240isin3\theta cos2\theta\\ -13440isin3\theta cos\theta-240i\,sin3\theta\\ -240icos4\theta sin\theta+13440icos3\theta sin\theta\\ +30240isin\theta cos2\theta\\ +13440icos\theta sin\theta+240isin\theta\\ +120\,sin^2 4\theta+6720sin4\theta sin3\theta\\ +15120sin4\theta sin2\theta\\ +6720sin4\theta sin\theta-240sin4\theta sin3\theta\\ -13440sin^2 3\theta-30240sin3\theta sin2\theta-13440sin3\theta sin\theta\\ +240sin4\theta sin\theta\\ +13440sin3\theta sin\theta\\ +30240sin2\theta sin\theta+6720sin^2\theta\end{array}\right]}{\left[\begin{array}{c}cos^2 4\theta+56cos4\theta cos3\theta+126cos4\theta cos2\theta+56cos4\theta cos\theta+cos4\theta\\ +56cos4\theta cos3\theta+3136cos^2 3\theta+7056cos3\theta cos2\theta\theta+3136cos3\theta cos\theta+56cos3\theta\\ +126cos2\theta cos4\theta+7056cos2\theta+15876cos^2 2\theta+7056cos2\theta cos\theta\\ +126cos2\theta+56cos4\theta cos\theta+3136cos3\theta cos\theta+7056cos2\theta\theta cos\theta\theta+3136cos^2\theta+56cos\theta+cos4\theta\\ +56cos3\theta+126cos2\theta+56cos\theta+1+sin^2 4\theta+56sin4\theta sin3\theta+126sin4\theta sin2\theta+56sin4\theta sin\theta\\ +56sin4\theta sin3\theta+3136sin^2 3\theta+7056sin3\theta sin2\theta+3136sin3\theta sin\theta\\ +126sin4\theta sin2\theta+7056sin2\theta sin3\theta+15876sin^2 2\theta+7056sin2\theta sin\theta+56sin4\theta sin\theta\\ +3136sin3\theta sin\theta+7056sin2\theta sin\theta+3136sin^2\theta\end{array}\right]}.$$

Since we are interested only on the real part, then we collect like terms only on the real part as follows:

$$X(\theta)=\frac{120\left[\begin{array}{c}cos^2 4\theta+54cos4\theta cos3\theta+126cos4\theta cos2\theta+58cos4\theta cos\theta\\ -112cos^2 3\theta-252cos3\theta cos2\theta-58cos\theta+252cos\theta cos2+1125cos^2\theta\\ -54cos\theta\theta-126cos2\theta-1+sin^2 4\theta+54sin4\theta sin3\theta+126sin4\theta sin2\theta\\ +58sin^2 4\theta-112sin^2 3\theta-252sin3\theta sin2\theta+252sin2\theta sin\theta\theta+56sin^2\theta\end{array}\right]}{\left[\begin{array}{c}cos^2 4\theta+112cos4\theta cos3\theta+252cos4\theta cos2\theta+112cos4\theta cos\theta\\ +2cos4\theta+3136cos^2 3\theta+14112cos2\theta cos\theta+6272cos3\theta cos\theta\\ +112cos3\theta+15876cos^2 2\theta+14112cos2\theta cos\theta+112cos\theta\\ +126cos2\theta+1+sin^2 4\theta+112sin4\theta cos3\theta+252sin4\theta sin2\theta\\ +112sin4\theta sin\theta+3136sin^2 3\theta+14112sin3\theta sin2\theta+6272sin3\theta sin\theta\\ +15876sin^2 2\theta+14112sin2\theta sin\theta+3136sin^2\theta\end{array}\right]}$$

We now evaluate $X(\theta)$ for the interval $0\le\theta\le 180$ as follows:

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For $\theta = 0$ we have

$$X(\theta) = \frac{120\begin{bmatrix} cos^2 0 + 54cos0cos0 + 126cos0cos0 \\ +58cos0cos0 - 112cos^2 0 - 252cos0cos0 \\ -58cos0 + 252cos0cos0 + 1125cos^2 0 \\ -54cos0 - 126cos0 - 1 + sin^2 0 \\ +54sin0sin0 + 126sin0sin0 + 58sin^2 0 - 112sin^2 0 \\ -252sin0sin0 + 252sin0sin0 + 56sin^2 0 \end{bmatrix}}{\begin{bmatrix} cos^2 0 + 112cos0cos0 + 252cos0cos0 + 112cos0cos\theta \\ +2cos0 + 3136cos^2 0 \\ +14112cos0cos0 + 6272cos0cos0 + 112cos0 + 15876cos^2 0 \\ +14112cos0cos0 + 112cos0 + 126cos0 + 1 + sin^2 0 \\ +112sin0cos0 \\ +252sin0sin0 + 112sin0sin0 + 3136sin^2 0 + 14112sin0sin0 \\ +6272sin0sin0 + 15876sin^2 0 + 14112sin0sin0 + 3136sin^2 0 \end{bmatrix}}$$

$$X(\theta) = \frac{120\begin{bmatrix} 1 + 54 + 126 + 58 - 112 - 252 - 58 + 252 \\ +112 - 54 - 126 - 1 + 0 + 0 + 0 + 0 - 0 - 0 + 0 + 0 \end{bmatrix}}{\begin{bmatrix} 1 + 112 + 252 + 112 + 2 + 3136 + 14112 + 6272 + 1 + 112 + 252 \\ +112 + 2 + 3136 + 14112 \\ +6272 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 15876 \end{bmatrix}}$$

Therefore

$X(\theta) = 0.$

And for $\theta = 180$ we have

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$$X(\theta)=\frac{120\left[\begin{array}{c} cos^2 720+54cos720cos540+126cos720cos360 \\ +58cos720cos180-112cos^2 540-252cos540cos360-58cos180 \\ +252cos180cos360+1125cos^2 180-54co180-126cos360 \\ -1125cos180-1+sin^2 720+54sin720sin540 \\ +126sin720sin360+58sin^2 720-112sin^2 540+252sin540sin360 \\ +252sin360sin180+56sin^2 180 \end{array}\right]}{\left[\begin{array}{c} cos^2 720+112cos720cos540+252cos720cos360 \\ +112cos720cos180+2cos720+3136cos^2 540+14112cos360cos180 \\ +6272cos540cos180+112cos540+15876cos^2 360+14112cos360cos180 \\ +112cos180+126cos360+1+sin^2 720+112sin720cos540 \\ +252sin720sin360+112sin720sin180+3136sin^2 540 \\ +14112sin540sin360+6272sin540sin180+15876sin^2 360+14112sin360sin180 \\ +3136sin^2 180 \end{array}\right]}$$

Therefore

$X(\theta) \cong -5.$

Therefore our four step method has an interval of absolute stability $[-5,0]$.

4. CONCLUSION

The developed four-step linear multi-step method stands as a noteworthy solution for the numerical approximation of third-order ordinary differential equations. Its consistency, zero stability, and convergence properties validate its reliability and effectiveness. The substantial interval of absolute stability further enhances its applicability to a wide range of dynamic systems. Notably, the derivation technique employed in this work is characterized by its simplicity and adaptability, setting this method apart from traditional collocation-based approaches. This research offers a valuable contribution to numerical analysis, providing a practical and efficient tool for solving complex ODEs in scientific and engineering applications.

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References

[1] Ismail, M. S., El-Tawil, M. A., & El-Danaf, T. S. (2014). On the Construction of Linear Multistep Methods.*Abstract and Applied Analysis, 2014,* 1-7.

[2] Burden, R. L., & Faires, J. D. (2016). Numerical Analysis.*Cengage Learning.*

[3] Butcher, J. C. (2008). Numerical Methods for Ordinary Differential Equations.*John Wiley & Sons.*

[4] Ascher, U. M., & Petzold, L. R. (1998). Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations.*SIAM.*

[5] Hairer, E., Nørsett, S. P., &Wanner, G. (1993). Solving Ordinary Differential Equations I: Nonstiff Problems.*Springer-Verlag.*

[6] Lambert, J. D. (1973). Computational Methods in Ordinary Differential Equations.*John Wiley & Sons.*

[7] Shampine, L. F., & Gordon, M. K. (1975). Computer Solution of Ordinary Differential Equations: The Initial Value Problem.*W. H. Freeman and Company.*