IJO - INTERNATIONAL JOURNAL OF MATHEMATICS

(ISSN: 2992-4421) I.U. Udo-Akpan[1] *

https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

ON A-TWO-PARAMETER DYNAMIC BUCKLING OF A VISCOUSLY DAMPED BUT CLAMPED COLUMN STRESSED BY A STEP LOAD

I.U. Udo-Akpan[1] and J. U. Chukwuchekwa[2]

[1]*Department of Mathematics and Statistics, University of Port Harcourt, Port Harcourt, Rivers State, Nigeria.*

[2]*Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria.*

Abstract

In this investigation, we extend our search on the dynamic buckling loads of some elastic structures to that of a clamped column lying on a nonlinear (cubic) elastic foundation but impacted upon axially by a step load. In order to ensure a uniformly valid solution, we employ multi–scaling two–timing regular perturbation procedures in asymptotic expansions of the variables. It is shown that (a) clamped columns buckle at higher buckling loads than columns with simply–supported ends irrespective of whether the columns are loaded statically or dynamically and whether damped or undamped, (b) Specifically, the inequalities satisfied by the static buckling load λ_S and dynamic buckling loadλ_D in the clamped case are respectively given as $1 < \lambda_S < 2.125$ and $1 < \lambda_D < 2.125$ as against $0 < \lambda_S < 1$ and $0 < \lambda_D < 1$ for simply–supported end conditions, (c) At low values of the static buckling load λ_S, there is no appreciable change in the values of the dynamic buckling load λ_D but at higher values of λ_S, λ_D increases sharply with increased static buckling load λ_S. However, the increase seems to decrease with increased damping. We are able to mathematically relate the dynamic buckling load λ_D to the static buckling load λ_S and thereby by–passing the labour of repeating the entire process for different imperfection parameters. Thus, given either λ_D or λ_S, we can predict either value without the actual knowledge of the size of the small imperfection parameter.

Keywords: Dynamic buckling, viscously damped clamped column, elastic structures, step load, two-timing perturbation 2010 Mathematical Subject Classification: 74B20, 74H10, 34E10

1. INTRODUCTION

Investigations into the static or dynamic stability (or otherwise) of columns (finite or infinite) are age old problems that have been embarked upon by researchers for some years now. As observed by [1] and [2], buckling of structures is one of the structural instabilities that have been known for centuries ever since the equation to derive the critical buckling of a column was derived by Leonhard Euler [3]. As a result of this, previous studies on the subject matter are indeed enormous and varied, and include investigations by [4]-[7], among others. We must stress that columns, are in themselves, indispensable structural materials and their utility cuts across all cultures in human history.

This investigation is concerned with analytical determination of the dynamic buckling load of a viscously damped but clamped imperfect column that rests on a nonlinear (cubic) elastic foundation, where the column is struck by a step load. The viscous

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

damping, though small in magnitude, is not in any way related either physically or mathematically, to an equally small imperfection that is stress-free and twice-differentiable. The formulation therefore contains two small but dimensionally independent parameters upon which asymptotic expansions are initiated using a two-timing multi-scaling perturbation procedure.

The analysis contained here is an extension of a similar study espoused by [8], where, in that study, damping was taken to be related in some way, to the imperfection $\bar{W}(x)$, so that once the imperfection was fixed, the damping was equally fixed. However, we note, from physical reasoning, that damping need not in any way, be related to the imperfection in all probabilities. Similar studies were done by [9]-[12], among others.

Apart from addressing the phenomenon of viscous damping in a special fashion, this investigation is related, in spirit, to similar studies by [13]-[25].

2. FORMULATION OF THE PROBLEM

As in [6], the dimensional differential equation satisfied by the deflection $W(X,T)$ of a finite viscously damped column lying on a nonlinear (cubic) elastic foundation but struck by a load $P(T)$ is

$$m_0 W_{,TT} + QW_{,T} + EIW_{,XXXX} + 2P(T)W_{,XX} + k_1 W - \alpha k_3 W^3 = -2P(T)\frac{d^2\bar{W}}{dX^2}, T > 0, \quad (2.1a)$$

$$W(X,0) = W_{,T}(X,0) = 0, \quad (2.1b)$$

$$W = W_{,X} = 0, at\ X = 0, \pi, \quad T > 0 \quad (2.1c)$$

Here, m_0 is the mass per unit length, Q is the damping coefficient, EI is the bending stiffness, where E and I are the Young's modulus and the moment of inertia respectively. The nonlinear elastic foundation exerts a force per unit length given by $k_1 W - \alpha k_3 W^3$ on the column, where k_1 and k_3 are constants such that k_1>0, k_3>0, and α is the imperfection-sensitivity parameter which is such that for $\alpha = 1$, the nonlinear elastic foundation is said to "softening", whereas for α = -1, the foundation is said to be "hardening". We have here excluded all nonlinearities of $W(X,T)$ higher than the cubic and have also excluded all nonlinear derivatives of$W(X,T)$.

3. NON-DIMENSIONALIZATION OF THE GOVERNING EQUATION

We shall non-dimensionalize the equations (2.1a - c) by using the following non-dimensional quantities

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X, \quad w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W, \quad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \quad \epsilon\bar{w} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \bar{W}, \quad 2\delta = \frac{Q}{(m_0 k_1)^{\frac{1}{2}}},$$

$$t = \left(\frac{K_1}{m_0}\right)^{\frac{1}{2}} T, \quad 0 < \epsilon << 1, \quad 0 < \delta \ll 1; \quad 0 < \lambda < 2.125;$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

On introducing these non-dimensional quantities into (2.1a-c) and simplifying, we get

$$w_{,tt} + 2\delta w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w - \alpha w^3 = -2\epsilon\lambda f(t)\frac{d^2\overline{w}}{dx^2}, \; t > 0,$$
$$0 < x < \pi \qquad (3.1a)$$

$$w(x,0) = w_{,t}(x,0) = 0, \; 0 < x < \pi \qquad (3.1b)$$

$$w = w_{,x} = 0, at \; x = 0, \pi, \quad t > 0 \qquad (3.1c)$$

Here, a subscript following a comma indicates partial differentiation and $f(t)$ indicates the actual time dependence of the load having its magnitude as λ. In our case, $f(t)$ is a step load characterized by

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \qquad (3.2)$$

4. CLASSICAL BUCKLING LOAD, λ_C

The classical buckling load λ_C is the load that the associated linear perfect column buckles statically. The equations required are obtained from (3.1a) as

$$w_{,xxxx} + 2\lambda w_{,xx} + w = 0, \quad 0 < x < \pi \qquad (4.1a)$$

$$w = w_{,x} = 0, \; at \; x = 0, \pi. \qquad (4.1b)$$

We note that the deflection w at this stage depends only on$x$. To solve (4.1a,b), we let

$$w(x) = \sum_{n=1}^{\infty}(1 - Cos2nx)U_n \qquad (4.2)$$

On substituting (4.2) into (4.1a),multiplying by $Cos2mx$ for fixed$m$ and integrating form 0 to π, we see that for $n = m$, we get

$$(16m^4 - 8m^2\lambda + 1)U_m = 0 \qquad (4.3)$$

where, m is a fixed value of n.

According to [18], the condition for static buckling is

$$\frac{d\lambda}{dw} = 0 \qquad (4.4)$$

where, w is the displacement (or deflection). This gives the classical buckling load λ_C as

$$\lambda_C = \frac{16m^4 + 1}{8m^2} \qquad (4.5a)$$

The least value of λ_C is when $m = 1$ and in this case, we get

$$\lambda_C = \frac{17}{8} = 2.125 \qquad (4.5b)$$

In retrospect, a similar column with simply-supported end conditions that satisfy the same equation as (4.1a) but, instead of (4.1b), it would satisfy the conditions

$$w = w_{,xx} = 0, \; at \; x = 0, \pi \qquad (4.5c)$$

has the classical buckling load as $\lambda_C = 1$ which is different from (4.5a,b). Thus, a clamped column has a higher classical buckling load than the same column with simply – supported end conditions.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

5. STATIC BUCKLING LOAD, λ_S

This is the load that the column would require to buckle statically. The required differential equation is

$$w_{,xxxx} + 2\lambda w_{,xx} + w - \alpha w^3 = -2\epsilon\lambda \frac{d^2\bar{w}}{dx^2}, \quad 0 < x < \pi \qquad (5.1a)$$

$$w = w_{,x} = 0, \ at\ x = 0, \pi, \qquad (5.1b)$$

To determine the displacement (or deflection) $w(x)$ in (5.1a,b), we set $f(t) \equiv 1$and let

$\bar{w} = \bar{a}_m(1 - Cos2mx), \quad |\bar{a}_m| \ll 1$

Next, we let

$$\sum_{i=1}^{\infty} V^{(i)} \epsilon^i ; \quad V^{(i)} = V^{(i)}(x) \qquad (5.2)$$

By substituting (5.2) into (5.1a) and equating the coefficients of powers of $\epsilon^i, i = 1, 2, 3, \ldots,$ we get

$$O(\epsilon): NV^{(1)} \equiv V^{(1)}_{,xxxx} + 2\lambda V^{(1)}_{,xx} + V^{(1)} = -8\lambda m^2 \bar{a}_m Cos2mx \qquad (5.3)$$

$$O(\epsilon^2): NV^{(2)} = 0 \qquad (5.4)$$

$$O(\epsilon^3): \quad NV^{(3)} = \alpha\left(V^{(1)}\right)^3 \qquad (5.5)$$

etc.

$$V^{(i)}(x) = V^{(i)}_{,x} \ \ at\ x = 0, \pi \qquad (5.6)$$

We seek for the solutions of (5.3) – (5.6) by letting

$$V^{(i)}(x) = 2\sum_{i=1}^{\infty} V_n^{(i)} \sin^2 nx = \sum_{i=1}^{\infty}(1 - Cos2nx)V_n^{(i)} \qquad (5.7)$$

On substituting (5.7) into (5.3), we get

$$\sum_{n=1}^{\infty}\left[V_n^{(1)}(8n^2\lambda - 16n^4)Cos2nx + V_n^{(1)}(1 - Cos2nx)\right] = -8\lambda m^2 \bar{a}_m Cos2mx \ (5.8)$$

On multiplying (5.8) through by Cos2mx and integrating from 0 to π, we see that for $n = m$,

$$(16m^4 - 8m^2\lambda + 1)V_m^{(1)} = 8\lambda m^2 \bar{a}_m$$

This gives

$$V_m^{(1)} = \frac{8\lambda m^2 \bar{a}_m}{16m^4 - 8m^2\lambda + 1} = B \qquad (5.9a)$$

$$\therefore V^{(1)} = V_m^{(1)}(1 - Cos2mx) \qquad (5.9b)$$

On substituting (5.7) into (5.4), we easily get

$$V_m^{(2)} = 0 \qquad (5.10)$$

Next, we substitute (5.7) in (5.5) and use (5.9a,b) to get

$$\sum_{n=1}^{\infty}\left[V_n^{(3)}(8n^2\lambda - 16n^4)Cos2nx + V_n^{(3)}(1 - Cos2nx)\right] = \alpha\left(V^{(1)}\right)^3(1 - Cos2mx)^3$$

$$= \alpha\left(V^{(1)}\right)^3\left[\frac{5}{2} - \frac{15}{4}Cos2mx + \frac{3}{2}Cos4mx - \frac{1}{4}Cos6mx\right] (5.11)$$

We multiply (5.11) through by Cos2mx and integrate from 0 to π, to get

$$V_m^{(3)} = \frac{15\alpha B^3}{4(16m^4 - 8m^2\lambda + 1)} \qquad (5.12)$$

Next, we multiply (5.11) by Cos4mx and integrate from 0 to π to get

$$V_{2m}^{(3)} = \frac{-3B^3\alpha}{2(256m^4 - 16m^2\lambda + 1)} \quad (5.13)$$

Lastly, we multiply (5.11) by Cos6mx and integrate from 0 to π and get,

$$V_{3m}^{(3)} = \left(\frac{\alpha B^3}{1296m^4 - 36m^2\lambda + 1}\right) \quad (5.14)$$

Thus, we get

$$V^{(3)} = V_m^{(3)}(1 - Cos2mx) + V_{2m}^{(3)}(1 - Cos4mx) + V_{3m}^{(3)}(1 - Cos6mx) \quad (5.15)$$

So that

$$w(x) = \epsilon V_m^{(1)}(1 - Cos2mx) + \epsilon^3\Big[V_m^{(3)}(1 - Cos2mx) + V_{2m}^{(3)}(1 - Cos4mx) + V_{3m}^{(3)}(1 - Cos6mx)\Big] + \dots \quad (5.16)$$

Before determining the static buckling load, we need to evaluate (5.16) at $x = \frac{\pi}{2m}$. This is informed by the fact that eventually, we shall need to determine the associated dynamic problem at $x = \frac{\pi}{2m}$ (which also means finding the maximum of (5.16)). Such evaluation yields

$$w = 2\epsilon V_m^{(1)} + 2\epsilon^3\left(V_m^{(3)} - V_{3m}^{(3)}\right) + \dots \quad (5.17)$$

We can write (5.17) as

$$w = C_1\epsilon + C_3\epsilon^3 + \dots \quad (5.18)$$

where,

$$C_1 = 2V_m^{(1)}, \quad C_3 = 2\left(V_m^{(3)} - V_{3m}^{(3)}\right) \quad (5.19)$$

As in [18], the static buckling load is obtained by first reversing the series (5.18) in the form,

$$\epsilon = d_1 w + d_3 w^3 + \dots \quad (5.20)$$

On substituting for w from (5.18) in (5.20) and equating the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{C_1}, \quad d_3 = \frac{-C_3}{C_1^4} \quad (5.21)$$

The maximization (4.4) is now easily executed from (5.20) to yield

$$d_1 + 3d_3 w_a^2 = 0 \quad (5.22)$$

where, w_a is the value of w evaluated at static buckling. From (5.18), we get

$$w_a = \sqrt{\frac{-d_1}{3d_3}} = \frac{1}{\sqrt{3}}\left(\frac{C_1^3}{C_3}\right)^{\frac{1}{2}} \quad (5.23)$$

Next, we determine (5.20) at static buckling and get

$$\epsilon = d_1 w + d_3 w^3 + \dots = w_a(d_1 + d_3 w_a^2) = \frac{2}{3\sqrt{3}}\left(\frac{C_1}{C_3}\right)^{\frac{1}{2}} \quad (5.24)$$

On substituting in (5.24) for C_1 and C_3, we get

$$(16m^4 - 8m^2\lambda_S + 1)^{\frac{3}{2}} = 18\sqrt{5}m^2\alpha^{\frac{1}{2}}\bar{a}_m\epsilon\lambda_S\left[1 + \frac{4}{15}\left(\frac{16m^4 - 8m^2\lambda_S + 1}{1296m^4 - 36m^2\lambda_S + 1}\right)\right]^{\frac{1}{2}} \quad (5.25)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

where λ_S is the static buckling load. The least value of λ_S is obtained when $m = 1$, and for this, we get

$$(17 - 8\lambda_S)^{\frac{3}{2}} = 18\sqrt{5}\lambda_S \alpha^{\frac{1}{2}} \bar{a}_1 \epsilon \left[1 + \frac{4}{15}\left(\frac{17 - 8\lambda_S}{1297 - 3\lambda_S}\right)\right]^{\frac{1}{2}} \quad (5.26)$$

If it is required that the buckling mode be strictly in the shape of imperfection, then the results corresponding to (5.25) and (5.26) respectively become

$$(16m^4 - 8m^2\lambda_S + 1)^{\frac{3}{2}} = 18\sqrt{5}m^2\lambda_S \alpha^{\frac{1}{2}} \bar{a}_m \epsilon \quad (5.27)$$

and

$$(17 - 8\lambda_S)^{\frac{3}{2}} = 18\sqrt{5}\lambda_S \alpha^{\frac{1}{2}} \bar{a}_1 \epsilon \quad (5.28)$$

By way of comparison, we can perform a similar analysis on the same column but with simply-supported end conditions and the results corresponding to those of (5.25), (5.26), (5.27) and (5.28) are respectively given by

$$(m^4 - 2m^2\lambda_S + 1)^{\frac{3}{2}} = \frac{9}{2}m^2\lambda_S \bar{a}_m \epsilon \alpha^{\frac{1}{2}} \left[1 + \left(\frac{m^4 - 2m^2\lambda_S + 1}{81m^4 - 18m^2\lambda_S + 1}\right)\right]^{\frac{1}{2}} \quad (5.29)$$

$$(1 - \lambda_S)^{\frac{3}{2}} = \frac{9}{4\sqrt{2}}\lambda_S \epsilon \bar{a}_1 \alpha^{\frac{1}{2}} \left[1 + \left(\frac{1 - \lambda_S}{41 - 9\lambda_S}\right)\right]^{\frac{1}{2}} \quad (5.30)$$

$$(m^4 - 2m^2\lambda_S + 1)^{\frac{3}{2}} = \frac{9}{2}m^2\lambda_S \bar{a}_m \epsilon \alpha^{\frac{1}{2}} \quad (5.31)$$

and

$$(1 - \lambda_S)^{\frac{3}{2}} = \frac{9}{4\sqrt{2}}\lambda_S \epsilon \bar{a}_1 \alpha^{\frac{1}{2}} \quad (5.32)$$

The result (5.32) was first obtained by [6]. So far, we conclude that the static buckling load of structures largely depends, among other things, on the type of end constraints of the structures.

6. THE DYNAMIC PROBLEM

The associated dynamic problem follows from (3.1a) which we now recast as

$$w_{,tt} + 2\delta w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w - \alpha w^3$$
$$= -2\epsilon\lambda f(t)\frac{d^2\bar{w}}{dx^2}, t > 0, \; 0 < x < \pi \quad (6.1a)$$

$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \quad (6.1b)$$

$$w = w_{,x} = 0, at\ x = 0, \pi, \quad t > 0 \quad (6.1c)$$

Henceforth, we shall substitute for the step load $f(t)$ as in (3.2).

We let

$$\tau = \delta t, \quad (6.2a)$$

$$\hat{t} = t + \left(\frac{\mu_2(\tau)\epsilon^2 + \mu_3(\tau)\epsilon^3 + \cdots}{\delta}\right) \quad (6.2b)$$

where, $\mu_i(0) = 0, \quad i = 1, 2, 3, \ldots, \quad \mu_i = \mu_i(\tau)$

We stress that ϵ and δ are small and unrelated parameters. Thus, from (6.2a,b), we get

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} + \frac{\partial w}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial \tau}\frac{d\tau}{dt} + \frac{\partial w}{\partial \tau}\frac{d\tau}{dt}$$
$$= (1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)w_{,\hat{t}} + \delta w_{,\tau} \quad (6.3a)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

I.U. Udo-Akpan[1] *

where, $(\ldots)' = \frac{d(\ldots)}{d\tau}$ and a subscript after a comma indicates partial differentiation.

$$\therefore \frac{\partial^2 w}{\partial t^2} = (1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)^2 w_{,\hat{t}\hat{t}} + \delta^2 w_{,\tau\tau} + 2\delta(1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)w_{,\hat{t}\tau}$$

$$+ \delta(\mu_2''\epsilon^2 + \mu_3''\epsilon^3 + \cdots)w_{,\hat{t}} \qquad (6.3b)$$

On substituting (6.3a,b) into (6.1a) for $f(t) = 1$, we get

$$\left[(1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)^2 w_{,\hat{t}\hat{t}} + \delta^2 w_{,\tau\tau} + 2\delta(1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)w_{,\hat{t}\tau} + \delta(\mu_2''\epsilon^2 + \mu_3''\epsilon^3 + \cdots)w_{,\hat{t}}\right] + 2\delta\left[(1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)w_{,\hat{t}} + \delta w_{,\tau}\right]$$

$$+ w_{,xxxx} + 2\lambda f(t) w_{,xx} + w - \alpha w^3 = -2\epsilon\lambda f(t)\frac{d^2\bar{w}}{dx^2} \qquad (6.4)$$

Next, we adopt the asymptotic series

$$w = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} U^{(ij)}(x,\hat{t},\tau)\epsilon^i\delta^j \qquad (6.5)$$

and substitute same into (6.4), and afterwards, equate powers of $\epsilon^i\delta^j$ to get

$$O(\epsilon):\quad LU^{(10)} = U^{(10)}_{,\hat{t}\hat{t}} + U^{(10)}_{,xxxx} + 2\lambda U^{(10)}_{,xx} + U^{(10)} = -2\lambda f(t)\frac{d^2\bar{w}}{dx^2} \qquad (6.6)$$

$$O(\epsilon\delta):\quad LU^{(11)} = -2\left(U^{(10)}_{,\hat{t}\tau} + U^{(10)}_{,\hat{t}}\right) \qquad (6.7)$$

$$O(\epsilon\delta^2):\quad LU^{(12)} = -2\left(U^{(10)}_{,\hat{t}\tau} + U^{(10)}_{,\hat{t}}\right) - U^{(10)}_{,\tau\tau} \qquad (6.8)$$

$$O(\epsilon^2):\quad LU^{(20)} = 0 \qquad (6.9)$$

$$O(\epsilon^2\delta):\quad LU^{(21)} = -2\left(U^{(20)}_{,\hat{t}\tau} + U^{(20)}_{,\hat{t}}\right) \qquad (6.10)$$

$$O(\epsilon^2\delta^2):\quad LU^{(22)} = -2\left(U^{(21)}_{,\hat{t}\tau} + U^{(21)}_{,\hat{t}}\right) - U^{(20)}_{,\tau\tau} \qquad (6.11)$$

$$O(\epsilon^3):\quad LU^{(30)} = \alpha\left(U^{(10)}\right)^3 - 2\mu_2' U^{(10)}_{,\hat{t}\hat{t}} \qquad (6.12)$$

$$O(\epsilon^3\delta):\quad LU^{(31)} = 3\alpha\left(U^{(10)}\right)^2 U^{(11)} - 2\left(U^{30}_{,\hat{t}\tau} + U^{(30)}_{,\hat{t}}\right) - \mu_2'' U^{(10)}_{,\hat{t}} - 2\mu_2' U^{(10)}_{,\hat{t}} \qquad (6.13)$$

$$O(\epsilon^3\delta^2):\quad LU^{(32)} = 3\alpha\left[U^{(10)}\left(U^{(11)}\right)^2 + \left(U^{(10)}\right)^2 U^{(12)}\right] - U^{(30)}_{,\tau\tau}$$

$$-2\left(U^{(31)}_{,\hat{t}\tau} + \mu_2' U^{(11)}_{,\hat{t}\tau}\right) - \mu_2'' U^{(10)}_{,\hat{t}} - 2\left(U^{(31)}_{,\hat{t}} + \mu_2' U^{(11)}_{,\hat{t}} + U^{(30)}_{,\tau}\right) \qquad (6.14)$$

The initial conditions, which are evaluated at $\hat{t} = 0 = \tau$ are

$$U^{(ij)}(x,0,0) = 0, \quad i = 1,2,3,\ldots;\ j = 0,1,2,3,\ldots \qquad (6.15)$$

$$O(\epsilon):\quad U^{(10)}_{,\hat{t}}(x,0,0) = 0 \qquad (6.16)$$

$$O(\epsilon\delta):\quad U^{(11)}_{,\hat{t}}(x,0,0) + U^{(10)}_{,\tau}(x,0,0) = 0 \qquad (6.17)$$

$$O(\epsilon\delta^2):\quad U^{(12)}_{,\hat{t}}(x,0,0) + U^{(11)}_{,\tau}(x,0,0) = 0 \qquad (6.18)$$

In general, we have

$$U^{(1k)}_{,\hat{t}}(x,0,0) + U^{1(k-1)}_{,\tau}(x,0,0) = 0, \quad k = 1,2,3,\ldots \qquad (6.19)$$

$$O(\epsilon^2):\quad U^{(20)}_{,\hat{t}}(x,0,0) = 0$$

(6.20)

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

In general, we have

$$U_{,\hat{t}}^{(2k)}(x,0,0) + U_{,\tau}^{2(k-1)}(x,0,0) = 0, \quad k = 1,2,3,\dots \qquad (6.21)$$

$O(\epsilon^3)$: $$U_{,\hat{t}}^{(30)}(x,0,0) + \mu_2'(0)U_{,\hat{t}}^{(10)}(x,0,0) = 0 \qquad (6.22)$$

$O(\epsilon^3\delta)$: $$U_{,\hat{t}}^{(31)}(x,0,0) + \mu_2'(0)U_{,\hat{t}}^{(11)}(x,0,0) + U_{,\tau}^{(30)}(x,0,0) = 0 \qquad (6.23)$$

$O(\epsilon^3\delta^2)$: $$U_{,\hat{t}}^{32}(x,0,0) + \mu_2'(0)U_{,\hat{t}}^{(12)}(x,0,0) + U_{,\tau}^{(31)}(0,0) = 0 \qquad (6.24)$$

Generally, we have

$$U_{,\hat{t}}^{(3k)}(x,0,0) + \mu_2'(0)U_{,\hat{t}}^{(1k)}(x,0,0) + U_{,\tau}^{(3(k-1))}(x,0,0) = 0, k = 1,2,3,\dots \qquad (6.25)$$

The boundary conditions are

$$U^{(ij)} = U_{,x}^{(ij)} = 0 \text{ at } x = 0, \pi.$$

For solution to all the systems of equation involved here, we set

$$U^{(ij)}(x,\hat{t},\tau) = 2\sum_{n=1}^{\infty} U_n^{(ij)}(\hat{t},\tau)\sin^2 nx = \sum_{n=1}^{\infty} U_n^{(ij)}(\hat{t},\tau)(1-\cos 2nx) \qquad (6.26a)$$

We shall assume

$$\bar{w}(x) = \bar{a}_m(1-\cos 2mx) \qquad (6.26b)$$

for m, fixed.

On substituting (6.26a,b) into (6.6) and simplifying, we get

$$\sum_{n=1}^{\infty}\left(U_{n,\hat{t}\hat{t}}^{(10)} + U_n^{(10)}\right)(1-\cos 2mx) + \sum_{n=1}^{\infty}(8\lambda n^2 - 16n^4)U_n^{(10)}\cos 2nx = -8\lambda m^2\bar{a}_m cos2mx \qquad (6.27a)$$

Next, we multiply (6.27a) by $cos2mx$ and integrate from 0 to π , and note that for $n = m$, we get

$$U_{m,\hat{t}\hat{t}}^{(10)} + \varphi^2 U_m^{(10)} = 8\lambda m^2\bar{a}_m \qquad (6.27b)$$

$$U_m^{(10)}(0,0) = 0, \quad U_{m,\hat{t}}^{(10)}(0,0) = 0 \qquad (6.27c)$$ where,

$$\varphi^2 = (16m^4 - 8m^2\lambda + 1) > 0, \quad \forall\, m \qquad (6.27d)$$

The solution of (6.27b-d) is

$$U_m^{(10)}(\hat{t},\tau) = \beta_1(\tau)\cos\varphi\hat{t} + \theta_1(\tau)\sin\varphi\hat{t} + B \qquad (6.28a)$$

$$B = \frac{8\lambda m^2\bar{a}_m}{\varphi^2} = \frac{8\lambda m^2\bar{a}_m}{16m^4 - 8m^2\lambda + 1} \qquad (6.28b)$$

where,

$$\beta_1(0) = -B, \quad \theta_1(0) = 0 \qquad (6.28c)$$

This means that

$$U^{(10)} = U_m^{(10)}(1-\cos 2mx) \qquad (6.28d)$$

We next substitute (6.28d) into (6.7), using (6.25), and thereafter, multiply by $\cos 2mx$ and note that for $n = 2m$, we get

$$U_{m,\hat{t}\hat{t}}^{(11)} + \varphi^2 U_m^{(11)} = -2\left(U_{m,\hat{t}\tau}^{(10)} + U_{m,\hat{t}}^{(10)}\right) \qquad (6.29a)$$

$$U_m^{(11)}(0,0) = 0, \quad U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(10)}(0,0) \qquad (6.29b)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

On substituting for $U_m^{(10)}$ in (6.29) from (6.28a), we ensure a uniformly valid solution in $\hat{t}$ by equating to zero the coefficients of $\cos\varphi\hat{t}$ and $\sin\varphi\hat{t}$ and getting

$$\theta_1' + \theta_1 = 0, \quad \beta_1' + \beta_1 = 0 \qquad (6.30a)$$

The solution of (6.30a) using (6.28c) yields

$$\theta_1(\tau) = 0, \quad \beta_1(\tau) = -Be^{-\tau} \qquad (6.30b)$$

The remaining equation in (6.29a,b) is solved to get

$$U_m^{(11)} = \beta_2(\tau)\cos\varphi\hat{t} + \theta_2(\tau)\sin\varphi\hat{t} \qquad (6.31a)$$

$$\beta_2(0) = 0, \quad \theta_2(0) = \frac{B}{\varphi} \qquad (6.31b)$$

We however note that

$$\beta_1'(0) = B, \quad \beta_1''(0) = -B \qquad (6.31c)$$

We equally note at this stage that (6.32)

$$U^{(11)} = U_m^{(11)}(1 - \cos 2mx)$$

On substituting from (6.32) and (6.28d) in (6.8), multiplying thereafter by $\cos 2mx$ and integrating from 0 to π, we get, (for $n = m$)

$$U_{m,\hat{t}\hat{t}}^{(12)} + \varphi^2 U_m^{(12)} = -2\left(U_{m,\hat{t}\tau}^{(11)} + U_{m,\hat{t}}^{(11)}\right) - U_{m,\tau\tau}^{(10)} \qquad (6.33a)$$

$$U_m^{(12)}(0,0) = 0, \quad U_{m,\hat{t}}^{(12)}(0,0) + U_{m,\tau}^{(11)}(0,0) \qquad (6.33b)$$

Next, we substitute in (6.33a) for $U_m^{(11)}$ and $U_m^{(10)}$ from (6.31a) and (6.28a) and ensure a uniformly valid solution in $\hat{t}$ by equating to zero the coefficients of $\cos\varphi\hat{t}$ and $\sin\varphi\hat{t}$ and so, get, respectively

$$\theta_2' + \theta_2 = \frac{-\beta_1''}{2\varphi} \quad \text{and} \quad \beta_2' + \beta_2 = 0 \qquad (6.33c)$$

On solving (6.33c), we get

$$\theta_2(\tau) = e^{-\tau}\left[\int_0^{\tau}\frac{-\beta_1''(s)}{2\varphi}ds + \theta_2(0)\right], \quad \beta_2(\tau) = 0 \qquad (6.33d)$$

The remaining equation in the substitution into (6.33a) is solved to yield

$$U_m^{(12)}(\hat{t},\tau) = \beta_3(\tau)\cos\varphi\hat{t} + \theta_3(\tau)\sin\varphi\hat{t} \qquad (6.34a)$$

where,

$$\beta_3(0) = 0, \quad \theta_3(0) = 0 \qquad (6.34b)$$

In passing, we note from (6.33c) that

$$\theta_2'(0) = -\theta_2(0)\frac{-\beta_1''(\tau)}{2\varphi} = \frac{-3B}{2\varphi}, \quad \theta_2''(0) = \frac{B}{\varphi} \qquad (6.34c)$$

We conclude that

$$U^{(12)} = U_m^{(12)}(1 - \cos 2mx) \qquad (6.35)$$

On solving equations (6.9) to (6.11), using the appropriate initial and boundary conditions, we get

$$U^{(20)}(x,\hat{t},\tau) = U^{(21)}(x,\hat{t},\tau) = U^{(22)}(x,\hat{t},\tau) = 0 \qquad (6.36)$$

We next substitute on the right hand side of (6.12), using (6.25) and simplify to get

$$LU^{(30)} = \alpha\left[\left(B^3 + \frac{3\beta_1^2 B}{2}\right) + 3\left(B^2\beta_1 + \frac{\beta_1^3}{4}\right)\cos\varphi\,\hat{t} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4}\right]\left(\frac{5}{2} - \frac{15\cos 2mx}{4} + \frac{3}{2}\cos 4mx - \frac{1}{4}\cos 6mx\right)$$

$$-2\mu_2'\,U_{m,\hat{t}\hat{t}}^{(10)}(1 - \cos 2mx) \qquad (6.37)$$

Next, we assume

$$U^{(30)} = \sum_{n=1}^{\infty} U_n^{(30)}(1 - \cos 2nx) \qquad (6.38)$$

On substituting (6.38) into (6.37) and first multiplying through by $\cos 2mx$, and thereafter integrating from 0 to π, we see that, for $n = m$, we get

$$U_{m,\hat{t}\hat{t}}^{(30)} + \varphi^2 U_m^{(30)} = \frac{15\alpha}{4}\left[\left(B^3 + \frac{3\beta_1^2 B}{2}\right) + 3\left(B^2\beta_1 + \frac{\beta_1^3}{4}\right)\cos\varphi\,\hat{t} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4}\right] - 2\mu_2'\,U_{m,\hat{t}\hat{t}}^{(10)} \qquad (6.39a)$$

$$U_m^{(30)}(0,0) = 0, \quad U_{m,\hat{t}}^{(30)}(0,0) + \mu_2'(0)U_{m,\hat{t}}^{(10)}(0,0) = 0 \qquad (6.39b)$$

Next, if in the substitution in (6.12), we multiply through by $cos4mx$ and thereafter integrate from 0 to π, we get (for $n = 2m$)

$$U_{2m,\hat{t}\hat{t}}^{(30)} + \varphi_{2m}^2 U_{2m}^{(30)} = \frac{-3\alpha}{2}\left[\left(B^3 + \frac{3\beta_1^2 B}{2}\right) + 3\left(B^2\beta_1 + \frac{\beta_1^3}{4}\right)\cos\varphi\,\hat{t} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4}\right] \qquad (6.40a)$$

$$U_{2m}^{(30)}(0,0) = 0, \quad U_{2m,\hat{t}}^{(30)}(0,0) = 0 \qquad (6.40b)$$

where,

$$\varphi_{2m}^2 = (256m^4 - 32m^2\lambda + 1) > 0, \ \forall\, m \qquad (6.40c)$$

Lastly, if in the substitution into (6.12), we multiply through by $\cos 6mx$, and integrate from 0 to π, we see that, for $n = 3m$, we get

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \varphi_{3m}^2 U_{3m}^{(30)} = \frac{\alpha}{4}\left[\left(B^3 + \frac{3\beta_1^2 B}{2}\right) + 3\left(B^2\beta_1 + \frac{\beta_1^3}{4}\right)\cos\varphi\,\hat{t} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4}\right] \qquad (6.41a)$$

$$U_{3m}^{(30)}(0,0) = 0, \quad U_{3m,\hat{t}}^{(30)}(0,0) = 0 \qquad (6.41b)$$

where,

$$\varphi_{3m}^2 = (1296m^4 - 72m^2\lambda + 1) > 0, \ \forall\, m \qquad (6.41c)$$

To solve (6.39a,b), we substitute for $U_m^{(10)}$ from (6.28a), (noting (6.28c)), and ensure a uniformly valid solution in $\hat{t}$ by equating to zero, the coefficient of $\cos\varphi\,\hat{t}$ and get

$$\mu_2'(\tau) = \frac{-45\alpha}{8\varphi}\left(B^2 + \frac{\beta_1^2}{4}\right), \quad \mu_2'(0) = \frac{-225\alpha B^2}{32\varphi^2} \qquad (6.42a)$$

$$\mu_2''(0) = \frac{45\alpha B^2}{16\varphi^2} \qquad (6.42b)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

The remaining equation in (6.39a) is now solved to get

$$U_m^{(30)}(\hat{t},\tau) = \beta_4 \cos\varphi\,\hat{t} + \Theta_4 \sin\varphi\,\hat{t} + \frac{15\alpha}{4}\left[\frac{r_0}{\varphi^2} - \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2\varphi^2} - \frac{\beta_1^3\cos 3\varphi\hat{t}}{32\varphi^2}\right] \qquad (6.43a)$$

where

$$r_0(\tau) = \left(B^3 + \frac{3\beta_1^2 B}{2}\right) \qquad (6.43b)$$

$$\beta_4(0) = \frac{5\alpha B^3}{128\varphi^2}, \quad (0) = 0 \qquad (6.43c)$$

$$r_0(0) = \frac{5B^3}{2}, \quad r_0'(0) = -3B^3, \quad r_0''(0) = 6B^3 \qquad (6.43d)$$

Turning to (6.40a,b), we solve to get

$$U_{2m}^{(30)}(\hat{t},\tau) = \beta_5 \cos\varphi_{2m}\,\hat{t} + \Theta_5 \sin\varphi_{2m}\,\hat{t} - \frac{3\alpha}{2}\left[\frac{r_0}{\varphi_{2m}^2} + \frac{r_1\cos\varphi\hat{t}}{\varphi_{2m}^2-\varphi^2} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2(\varphi_{2m}^2-4\varphi^2)} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4(\varphi_{2m}^2-9\varphi^2)}\right] \qquad (6.44a)$$

where,

$$r_1(\tau) = 3\left(B^2\beta_1 + \frac{\beta_1^3}{4}\right), \quad r_1(0) = \frac{-15B^3}{4}, \quad r_1'(0) = \frac{21B^3}{4}, \quad r_1''(0) = \frac{-39B^3}{4} \qquad (6.44b)$$

$$\beta_5(0) = \frac{3\alpha F_3 B^3}{4}, \quad \Theta_5(0) = 0 \qquad (6.44c)$$

$$F_3 = \frac{1}{2\varphi_{2m}^2} + \frac{15}{2(\varphi_{2m}^2-4\varphi^2)} - \frac{3}{\varphi_{2m}^2-4\varphi^2} + \frac{1}{2(\varphi_{2m}^2-9\varphi^2)} \qquad (6.44d)$$

We next solve (6.41a) and get

$$U_{3m}^{(30)}(\hat{t},\tau) = \beta_6 \cos\varphi_{3m}\,\hat{t} + \Theta_6 \sin\varphi_{3m}\,\hat{t} + \frac{\alpha}{4}\left[\frac{r_0}{\varphi_{3m}^2} + \frac{r_1\cos\varphi\hat{t}}{\varphi_{3m}^2-\varphi^2} + \frac{3B\beta_1^2\cos 2\varphi\hat{t}}{2(\varphi_{3m}^2-4\varphi^2)} + \frac{\beta_1^3\cos 3\varphi\hat{t}}{4(\varphi_{3m}^2-9\varphi^2)}\right] \qquad (6.45a)$$

where,

$$\beta_6(0) = \frac{\alpha B^3 F_4}{4}, \quad F_4 = \left[\frac{1}{\varphi_{3m}^2} + \frac{15}{\varphi_{3m}^2-\varphi^2} - \frac{3}{\varphi_{3m}^2-4\varphi^2} + \frac{1}{2(\varphi_{3m}^2-9\varphi^2)}\right], \quad \Theta_6(0) = 0 \qquad (6.45b)$$

Thus, we have

$$U^{(30)} = U_m^{(30)}(1-\cos 2mx) + U_{2m}^{(30)}(1-\cos 4mx) + U_{3m}^{(30)}(1-\cos 6mx) \qquad (6.45c)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

We next substitute on the right hand side of (6.13) and get

$$LU^{(31)} = \frac{3\alpha}{4}\left[\theta_2\left\{\left(B^2+\frac{\beta_1^2}{4}\right)\sin\varphi\,\hat{t} + B\beta_1\sin 2\varphi\,\hat{t} - \frac{\beta_1^2}{4}\sin 3\varphi\,\hat{t}\right\}\left\{\frac{5}{2} - \frac{15\cos 2mx}{4} + \frac{3\cos 4mx}{2} - \frac{\cos 6mx}{4}\right\}\right] - 2\left[\mathrm{U}^{(30)}_{m,\hat{t}\tau}(1-\cos 2mx) + \mathrm{U}^{(30)}_{2m,\hat{t}\tau}(1-\cos 4mx) + \mathrm{U}^{(30)}_{3m,\hat{t}\tau}(1-\cos 6mx) + \mathrm{U}^{(30)}_{m,\hat{t}}(1-\cos 2mx) + \mathrm{U}^{(30)}_{2m,\hat{t}}(1-\cos 4mx) + \mathrm{U}^{(30)}_{3m,\hat{t}}(1-\cos 6mx)\right] - \mu_2''\,\mathrm{U}^{(10)}_{m,\hat{t}}(1-\cos 2mx)$$

$$-2\mu_2'\,\mathrm{U}^{(10)}_{m,\hat{t}}(1-\cos 2mx) \qquad (6.46)$$

Let

$$U^{(31)} = \sum_{n=1}^{\infty}\mathrm{U}^{(31)}_{n}(1-\cos 2nx) \qquad (6.47)$$

On substituting (6.47) into (6.46), first multiplying through by $\cos 2mx$ and integrating from 0 to π, we get, for $n = m$

$$\mathrm{U}^{(31)}_{m,\hat{t}\hat{t}} + \varphi^2\mathrm{U}^{(31)}_{m} = \frac{45\alpha}{4}\left[\theta_2\left\{\left(B^2+\frac{\beta_1^2}{4}\right)\sin\varphi\,\hat{t} + B\beta_1\sin 2\varphi\,\hat{t} - \frac{\beta_1^2}{4}\sin 3\varphi\,\hat{t}\right\}\right]$$

$$-\mu_2''\,\mathrm{U}^{(10)}_{m,\hat{t}} - 2\mu_2'\,\mathrm{U}^{(10)}_{m,\hat{t}} - 2\left(\mathrm{U}^{(30)}_{2m,\hat{t}\tau} + \mathrm{U}^{(30)}_{2m,\hat{t}}\right) \qquad (6.48a)$$

$$\mathrm{U}^{(31)}_{m}(0,0) = 0, \quad U^{(31)}_{m,\hat{t}}(0,0) + \mu_2'(0)U^{(11)}_{m,\hat{t}}(0,0) + U^{(30)}_{m,\tau}(0,0) = 0 \qquad (6.48b)$$

Next, we multiply (6.46) by $\cos 4mx$, integrate from 0 to π, we get, for $n = 2m$

$$\mathrm{U}^{(31)}_{2m,\hat{t}\hat{t}} + \varphi_{2m}^2\mathrm{U}^{(31)}_{2m} = \frac{-9\alpha\theta_2}{8}\left[\left\{\left(B^2+\frac{\beta_1^2}{4}\right)\sin\varphi\,\hat{t} + B\beta_1\sin 2\varphi\,\hat{t} - \frac{\beta_1^2}{4}\sin 3\varphi\,\hat{t}\right\}\right]$$

$$-2\left(\mathrm{U}^{(30)}_{2m,\hat{t}\tau} + \mathrm{U}^{(30)}_{2m,\hat{t}}\right) \qquad (6.49a)$$

$$\mathrm{U}^{(31)}_{2m}(0,0) = 0, \quad U^{(31)}_{2m,\hat{t}}(0,0) + U^{(30)}_{2m,\tau}(0,0) = 0 \qquad (6.49b)$$

Similarly, we multiply (6.46) by $\cos 6mx$, integrate from 0 to π, we get, for $n = 3m$

$$\mathrm{U}^{(31)}_{3m,\hat{t}\hat{t}} + \varphi_{3m}^2\mathrm{U}^{(31)}_{3m} = \frac{3\alpha\theta_2}{16}\left[\left\{\left(B^2+\frac{\beta_1^2}{4}\right)\sin\varphi\,\hat{t} + B\beta_1\sin 2\varphi\,\hat{t} - \frac{\beta_1^2}{4}\sin 3\varphi\,\hat{t}\right\}\right]$$

$$-2\left(\mathrm{U}^{(30)}_{2m,\hat{t}\tau} + \mathrm{U}^{(30)}_{2m,\hat{t}}\right) \qquad (6.50a)$$

$$\mathrm{U}^{(31)}_{3m}(0,0) = 0, \quad U^{(31)}_{3m,\hat{t}}(0,0) + U^{(30)}_{3m,\tau}(0,0) = 0 \qquad (6.50b)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

To solve (6.48a), we substitute for terms there and after, ensure a uniformly valid solution by equating to zero the coefficients of $\cos\varphi\,\hat{t}$ and $\sin\varphi\,\hat{t}$. This gives, for $\cos\varphi\,\hat{t}$

$$\Theta_4' + \Theta_4 = 0 \qquad (6.51a)$$

Forsin $\varphi\,\hat{t}$:

$$\beta_4' + \beta_4 = \frac{\beta_1}{2}(\mu_2'' + 2\mu_2') - \frac{45\alpha}{32\varphi}(B^2 + \beta_1^2)\Theta_2 \qquad (6.51b)$$

On solving (6.51a,b), we get

$$\Theta_4 = 0; \quad \beta_4 = e^{-\tau}\left[\beta_4(0) - \int_0^\tau e^s\left\{\frac{\beta_1}{2}(\mu_2'' + 2\mu_2') + \frac{45\alpha}{32\varphi}(B^2 + \beta_1^2)\Theta_2\right\}ds\right] \qquad (6.51c)$$

$$\beta_4'(0) = \frac{-1545\alpha B^3}{128\varphi^2} \qquad (6.52d)$$

The remaining equations, having ensured a uniformly valid solution in (6.48a) are

$$\mathrm{U}_{m,\hat{t}\hat{t}}^{(31)} + \varphi^2\mathrm{U}_m^{(31)} = r_3\sin 2\varphi\,\hat{t} + r_4\sin 3\varphi\,\hat{t} \qquad (6.52a)$$

$$\mathrm{U}_m^{(31)}(0{,}0) = 0, \quad U_{m,\hat{t}}^{(31)}(0,0) + \mu_2'(0)U_{m,\hat{t}}^{(11)}(0,0) + U_{m,\tau}^{(30)}(0,0) = 0 \qquad (6.52b)$$

where,

$$r_3 = \frac{45\alpha\Theta_2 B\beta_1}{16} - \frac{15\alpha B}{2\varphi}\{(\beta_1^2)' + (\beta_1^2)\} \qquad (6.52c)$$

$$r_4 = \frac{-45\alpha\Theta_2 B\beta_1^2}{64} - \frac{15\alpha}{32\varphi}\{(\beta_1^3)' + (\beta_1^3)\} \qquad (6.52d)$$

$$(0) = \frac{75\alpha B^3}{16\varphi}, \quad r_4(0) = \frac{-225\alpha B^3}{64\varphi} \qquad (6.52e)$$

We note the following

$$\mathrm{r}_2'(0) = \frac{-345\alpha B^3}{128\varphi}, \quad \mathrm{r}_3'(0) = \frac{465\alpha B^3}{32\varphi}, \quad \mathrm{r}_4'(0) = \frac{-45\alpha B^3}{128\varphi} \qquad (6.52f)$$

On solving (6.52a,b), we get

$$\mathrm{U}_m^{(31)} = \beta_7\cos\varphi\,\hat{t} + \Theta_7\sin\varphi\,\hat{t} - \frac{r_3\sin 2\varphi\,\hat{t}}{3\varphi^2} - \frac{r_4\sin 3\varphi\,\hat{t}}{8\varphi^2} \qquad (6.53a)$$

where,

$$\beta_7(0) = 0 \qquad (6.53b)$$

$$\varphi\Theta_7(0) - \frac{2r_3(0)}{3\varphi} - \frac{3r_4(0)}{8\varphi} + \mu_2(0)\varphi\Theta_2(0) + \frac{15\alpha}{4}\left[\frac{r_0'}{\varphi^2} - \frac{B\beta_1'\beta_1}{\varphi^2} - \frac{3\beta_1^2\beta_1'}{\varphi^2}\right]\Big|_{\tau=0} = 0$$

This yields

$$\Theta_7(0) = \frac{-5695B^3}{512\varphi^3} \qquad (6.53c)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

To ensure a uniformly valid solution in terms of $\hat{t}$ in (6.49a), we equate to zero the coefficients of $\cos 2\varphi_{2m}\hat{t}$ and $\sin 2\varphi_{2m}\hat{t}$ and respectively get

$$\Theta_5' + \Theta_5 = 0 \qquad (6.53d)$$

and

$$\beta_5' + \beta_5 = \frac{\alpha}{2\varphi_{2m}}\left[\frac{9\,\Theta_2}{8}\left(B^2 + \frac{\beta_1^2}{4}\right) + \frac{3\varphi(r_1' + r_1)}{\varphi_{2m}^2 - \varphi^2}\right] \qquad (6.53e)$$

On solving (6.53d, e), we get

$$\Theta_5 = 0, \quad \beta_5 = e^{-\tau}\left[\beta_5(0) + \int_0^{\tau} e^{s} F_6\, ds\right] \qquad (6.53f)$$

where,

$$F_6 = \frac{\alpha}{2\varphi_{2m}}\left[\frac{9\,\Theta_2}{8}\left(B^2 + \frac{\beta_1^2}{4}\right) + \frac{3\varphi(r_1' + r_1)}{\varphi_{2m}^2 - \varphi^2}\right] \qquad (6.53g)$$

From (6.53e), we get

$$\beta_5'(0) = \alpha B^3 F_0, \quad F_0 = \frac{1}{2\varphi_{2m}}\left\{\frac{45}{32\varphi} + \frac{9\varphi}{2(\varphi_{2m}^2 - \varphi^2)}\right\} - \frac{3F_3}{4} \qquad (6.53h)$$

The remaining equation in (6.49a) is now written as

$$U_{2m,\hat{t}\hat{t}}^{(31)} + \varphi_{2m}^2 U_{2m}^{(31)} = r_5 \sin\varphi\,\hat{t} + r_6 \sin 2\varphi\,\hat{t} + r_7 \sin 3\varphi\,\hat{t} \qquad (6.54a)$$

$$U_{2m}^{(31)}(0,0) = 0, \quad U_{2m,\hat{t}}^{(31)}(0,0) + U_{2m,\tau}^{(30)}(0,0) = 0 \qquad (6.54b)$$

where,

$$r_5 = \frac{-9\alpha\Theta_2}{8}\left(B^2 + \frac{\beta_1^2}{4}\right) + \frac{3\alpha\varphi(r_1' + r_1)}{\varphi_{2m}^2 - \varphi^2} \qquad (6.54c)$$

$$r_6 = \frac{-9\alpha\Theta_2 B\beta_1}{8} - \frac{9\alpha\varphi B\left((\beta_1^2)' + (\beta_1^2)\right)}{\varphi_{2m}^2 - 4\varphi^2} \qquad (6.54d)$$

$$r_7 = \frac{9\alpha\beta_1^2\Theta_2}{32} - \frac{9\alpha\varphi\left((\beta_1^3)' + (\beta_1^3)\right)}{4(\varphi_{2m}^2 - 4\varphi^2)} \qquad (6.54e)$$

where,

$$r_5(0) = \alpha B^3 F_8, \qquad F_8 = \frac{-45}{32\varphi} + \frac{9\varphi}{2(\varphi_{2m}^2 - \varphi^2)} \qquad (6.54f)$$

$$r_6(0) = \alpha B^3 F_9, \qquad F_9 = 9\left(\frac{1}{8\varphi} + \frac{\varphi}{\varphi_{2m}^2 - 4\varphi^2}\right) \qquad (6.54g)$$

$$r_7(0) = \alpha B^3 F_{10}, \qquad F_{10} = \frac{9}{32\varphi} - \frac{9\varphi}{2(\varphi_{2m}^2 - 9\varphi^2)} \qquad (6.54h)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

On solving (6.54a,b), we get

$$U_{2m}^{(31)} = \beta_8 \cos\varphi_{2m}\hat{t} + \Theta_8 \sin\varphi_{2m}\hat{t} + \frac{r_5 \sin\varphi\hat{t}}{\varphi_{2m}^2 - \varphi^2} + \frac{r_6 \sin 2\varphi\hat{t}}{\varphi_{2m}^2 - 4\varphi^2} + \frac{r_7 \sin 3\varphi\hat{t}}{\varphi_{2m}^2 - 9\varphi^2} \qquad (6.55a)$$

$$\beta_8(0) = 0, \quad \Theta_8(0) = -\frac{1}{2\varphi_{2m}}\left[\frac{\varphi r_5}{\varphi_{2m}^2 - \varphi^2} + \frac{2\varphi r_6}{\varphi_{2m}^2 - 4\varphi^2} + \frac{3\varphi r_7}{\varphi_{2m}^2 - 9\varphi^2} + \beta_5'(0)\right.$$

$$\left.\left. = -\frac{3\alpha}{2}\left\{\frac{r_0'}{\varphi_{2m}^2} + \frac{r_1'}{\varphi_{2m}^2 - \varphi^2} + \frac{3B\beta_1'\beta_1}{\varphi_{2m}^2 - 4\varphi^2} + \frac{3\beta_1^2\beta_1}{4(\varphi_{2m}^2 - 9\varphi^2)}\right\}\right]\right|_{\tau=0} \qquad (6.55b)$$

That is

$$\Theta_8(0) = \alpha B^3 F_{11}, \qquad (6.55c)$$

$$F_{11} = -\frac{1}{2\varphi_{2m}}\left[\varphi\left\{\frac{F_8}{\varphi_{2m}^2 - \varphi^2} + \frac{2F_9}{\varphi_{2m}^2 - 4\varphi^2} + \frac{3F_{10}}{\varphi_{2m}^2 - 9\varphi^2}\right\}\right.$$

$$\left.+\frac{3}{2}\left\{\frac{1}{\varphi_{2m}^2} - \frac{21}{4(\varphi_{2m}^2 - \varphi^2)} + \frac{3}{\varphi_{2m}^2 - 4\varphi^2} - \frac{3}{\varphi_{2m}^2 - 9\varphi^2}\right\}\right] \qquad (6.55d)$$

Next we ensure a uniformly valid solution in $\hat{t}$ in (6.50a) by first substituting for the relevant terms there, and equating to zero the coefficients of $\cos\varphi_{3m}\hat{t}$ and $\sin\varphi_{3m}\hat{t}$ to get

$$\Theta_6' + \Theta_6 = 0, \quad \beta_6' + \beta_6 = 0 \qquad (6.56a)$$

On solving (6.56a), we get

$$\Theta_6(\tau) = 0, \quad \beta_6(\tau) = \beta_6(0)e^{-\tau} \qquad (6.56b)$$

where,

$$\beta_6'(0) = -\beta_6(0), \quad \beta_6''(0) = \beta_6(0) \qquad (6.56c)$$

The remaining equation in (6.50a) is now written as

$$U_{3m,\hat{t}\hat{t}}^{(31)} + \varphi_{2m}^2 U_{3m}^{(31)} = r_8 \sin\varphi\,\hat{t} + r_9 \sin 2\,\varphi\,\hat{t} + r_{10} \sin 3\,\varphi\,\hat{t} \qquad (6.57a)$$

$$U_{3m}^{(31)}(0,0) = 0, \quad U_{3m,\hat{t}}^{(31)}(0,0) + U_{3m,\tau}^{(30)}(0,0) = 0 \qquad (6.57b)$$

where,

$$r_8 = \frac{3\alpha\Theta_2}{16}\left(B^2 + \frac{\beta_1^2}{4}\right) + \frac{3\alpha\varphi\left(B^2\beta_1' + \frac{3\beta_1^2\beta_1'}{4}\right)}{2(\varphi_{3m}^2 - \varphi^2)} \qquad (6.57c)$$

$$r_9 = \frac{3\alpha\Theta_2 B\beta_1}{16} + \frac{3B\alpha\varphi(\beta_1'\beta_1 + \beta_1^2)}{2(\varphi_{3m}^2 - 4\varphi^2)} \qquad (6.57d)$$

$$r_{10} = \frac{-3\alpha\beta_1^2\Theta_2}{56} + \frac{3\alpha\varphi}{8}(\beta_1^2\beta_1' + \beta_1^3) \qquad (6.57e)$$

$$r_8(0) = \alpha B^3 F_7, \quad F_7 = \frac{15}{64\varphi} + \frac{3\varphi}{4(\varphi_{3m}^2 - \varphi^2)} \qquad (6.57f)$$

$$r_9(0) = \frac{-3\alpha B^3}{16\varphi}, \quad r_{10}(0) = \frac{-3\alpha B^3}{56} \qquad (6.57g)$$

$$r_8'(0) = \frac{-57\alpha B^3}{128\varphi} - \frac{9\varphi\alpha B^3}{4(\varphi_{3m}^2 - \varphi^2)} \qquad (6.57h)$$

$$r_9'(0) = \alpha B^3\left(\frac{15}{32\varphi} - \frac{3\varphi}{\varphi_{3m}^2 - \varphi^2}\right) \qquad (6.57i)$$

$$r_{10}'(0) = \frac{3\alpha B^3}{16\varphi} \qquad (6.57j)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

On solving (6.57a,b), we get

$$U_{3m}^{(31)} = \beta_9 \cos\varphi_{3m}\hat{t} + \Theta_9 \sin\varphi_{3m}\hat{t} + \frac{r_8 \sin\varphi\hat{t}}{\varphi_{3m}^2 - \varphi^2} + \frac{r_9 \sin 2\varphi\hat{t}}{\varphi_{3m}^2 - 4\varphi^2} + \frac{r_{10} \sin 3\varphi\hat{t}}{\varphi_{3m}^2 - 9\varphi^2} \quad (6.58a)$$

with

$$\beta_9(0) = 0, \quad (6.58b)$$

$$\Theta_9(0) = -\frac{1}{\varphi_{3m}}\left[\varphi\left\{\frac{r_8}{\varphi_{3m}^2 - \varphi^2} + \frac{2r_9}{\varphi_{3m}^2 - 4\varphi^2} + \frac{3r_{10}}{\varphi_{3m}^2 - 9\varphi^2}\right\} + \beta_6'\right.$$

$$\left.\left. + \frac{\alpha}{4}\left\{\frac{r_0'}{\varphi_{3m}^2} + \frac{r_1'}{\varphi_{3m}^2 - \varphi^2} + \frac{3B(\beta_1^2)'\beta_1}{2(\varphi_{3m}^2 - 4\varphi^2)} + \frac{3\beta_1^2\beta_1'}{4(\varphi_{3m}^2 - 9\varphi^2)}\right\}\right]\right|_{\tau=0} \quad (6.58c)$$

We note from (6.56a) that

$$\beta_6'(0) = -\beta_6(0) = \frac{-\alpha B^3 F_4}{8} \quad (6.58d)$$

On simplifying (6.58c), we get

$$\Theta_9(0) = \alpha B^3 S_1, \quad (6.58e)$$

$$S_1 = -\frac{1}{\varphi_{3m}}\left[\varphi\left\{\frac{F_7}{\varphi_{3m}^2 - \varphi^2} - \frac{3}{8(\varphi_{3m}^2 - 4\varphi^2)} - \frac{9}{56(\varphi_{3m}^2 - 9\varphi^2)}\right\} - \frac{F_4}{8}\right.$$

$$\left. + \frac{1}{4}\left\{\frac{3}{2\varphi_{3m}^2} + \frac{21}{4(\varphi_{3m}^2 - \varphi^2)} - \frac{3}{\varphi_{3m}^2 - 4\varphi^2} + \frac{3}{4(\varphi_{3m}^2 - 9\varphi^2)}\right\}\right] \quad (6.58f)$$

We next substitute on the right hand side of (6.14) and get

$$LU^{(32)} = 3\alpha\left[U_m^{(10)}\left(U_m^{(11)}\right)^2 + \left(U_m^{(10)}\right)^2 U_m^{(12)}\right](1 - \cos 2mx)$$

$$-\left[U_{m,\tau\tau}^{(30)}(1 - \cos 2mx) + U_{2m,\tau\tau}^{(30)}(1 - \cos 4mx) + U_{3m,\tau\tau}^{(30)}(1 - \cos 6mx)\right]$$

$$-2\left[U_{m,\hat{t}\tau}^{(31)}(1 - \cos 2mx) + U_{2m,\hat{t}\tau}^{(31)}(1 - \cos 4mx) + U_{3m,\hat{t}\tau}^{(31)}(1 - \cos 6mx)\right]$$

$$+\left[U_{m,\hat{t}}^{(31)}(1 - \cos 2mx) + U_{2m,\hat{t}}^{(31)}(1 - \cos 4mx) + U_{3m,\hat{t}}^{(31)}(1 - \cos 6mx)\right]$$

$$+ \mu_2' U_{m,\hat{t}\tau}^{(11)}(1 - \cos 2mx) - \mu_2'' U_{m,\hat{t}}^{(11)}(1 - \cos 2mx)$$

$$-2\left[\mu_2' U_{m,\hat{t}}^{(11)}(1 - \cos 2mx) + U_{m,\tau}^{(30)}(1 - \cos 2mx) + U_{2m,\tau}^{(30)}(1 - \cos 4mx)\right.$$

$$\left. + U_{3m,\tau}^{(30)}(1 - \cos 6mx)\right] \quad (6.59a)$$

By letting

$$U^{(32)} = \sum_{n=1}^{\infty} U_n^{(32)}(1 - \cos 2nx) \quad (6.59b)$$

and substituting same into (6.59a), multiplying the resultant equation by $\cos 2mx$ and integrating from 0 to π, we see that, for $n = m$, we get

$$U_{m,\hat{t}\hat{t}}^{(32)} + \varphi^2 U_m^{(32)} = \frac{45\alpha}{4}\left[U_m^{(10)}\left(U_m^{(11)}\right)^2 + \left(U_m^{(10)}\right)^2 U_m^{(12)}\right] - U_{m,\tau\tau}^{(30)}$$

$$- 2\left[U_{m,\hat{t}\tau}^{(31)} + U_{m,\hat{t}}^{(31)} + \mu_2' U_{m,\hat{t}\tau}^{(11)}\right] - \mu_2'' U_{m,\hat{t}}^{(11)} - 2\mu_2' U_{m,\hat{t}}^{(11)} - 2U_{m,\tau}^{(30)} \quad (6.60a)$$

$$U_m^{(32)}(0,0) = 0, \quad U_{m,\hat{t}}^{(32)}(0,0) + U_{m,\tau}^{(31)}(0,0) + \mu_2' U_{m,\hat{t}}^{(12)} = 0 \quad (6.60b)$$

For $n = 2m$ in the substitution into (6.59a), we get

$$U^{(32)}_{2m,\hat{t}\hat{t}} + \varphi^2_{2m} U^{(32)}_{2m} = \frac{-9\alpha}{4}\left[U^{(10)}_m\left(U^{(11)}_m\right)^2 + \left(U^{(10)}_m\right)^2 U^{(12)}_m\right] - U^{(30)}_{2m,\tau\tau}$$
$$- 2\left[U^{(31)}_{2m,\hat{t}\tau} + U^{(31)}_{2m,\hat{t}}\right] - 2U^{(30)}_{2m,\tau} \qquad (6.61a)$$

$$U^{(32)}_{2m}(0,0) = 0, \qquad U^{(32)}_{2m,\hat{t}}(0,0) + U^{(31)}_{2m,\tau}(0,0) \qquad (6.61b)$$

For $n = 3m$ in the substitution into (6.59a), we get

$$U^{(32)}_{3m,\hat{t}\hat{t}} + \varphi^2_{3m} U^{(32)}_{3m} = \frac{3\alpha}{4}\left[U^{(10)}_m\left(U^{(11)}_m\right)^2 + \left(U^{(10)}_m\right)^2 U^{(12)}_m\right] - U^{(30)}_{3m,\tau\tau}$$
$$-2\left[U^{(31)}_{3m,\hat{t}\tau} + U^{(31)}_{3m,\hat{t}}\right] - 2U^{(30)}_{3m,\tau} \qquad (6.62a)$$

$$U^{(32)}_{3m}(0,0) = 0, \qquad U^{(32)}_{3m,\hat{t}}(0,0) + U^{(31)}_{3m,\tau}(0,0) \qquad (6.62b)$$

If we substitute for terms on the right hand side of (6.60a), we get

$$U^{(32)}_{m,\hat{t}\hat{t}} + \varphi^2 U^{(32)}_m = \frac{45\alpha}{4}\left[\Theta_2^2\left\{\frac{\beta_1\cos\varphi\hat{t}}{4} + \frac{B}{2}(1-\cos 2\varphi\hat{t}) - \frac{\beta_1}{4}\cos 3\varphi\hat{t}\right\}\right.$$
$$+\left\{B\beta_1\beta_3 + \beta_3\left(B^2 + \tfrac{3\beta_1^2}{4}\right)\cos\varphi\hat{t} + \Theta_3\left(B^2 + \tfrac{\beta_1^2}{4}\right) sin\,\varphi\hat{t}\right.$$
$$\left.\left.+ B\beta_1\beta_3\cos 2\varphi\hat{t} + B\Theta_3\beta_1 sin\,2\varphi\hat{t} + \tfrac{\beta_3\beta_1^2}{4}\cos 3\varphi\hat{t} + \tfrac{\Theta_3\beta_1^2}{4}\sin 3\varphi\hat{t}\right\}\right]$$
$$-\left[\beta_4''\cos\varphi\hat{t} + \tfrac{15\alpha}{4}\left\{\tfrac{r_0''}{\varphi^2} - \tfrac{B(\beta_1^2)''}{2\varphi^2}\cos 2\varphi\hat{t} - \tfrac{(\beta_1^3)''}{32\varphi^2}\cos 3\varphi\hat{t}\right\}\right]$$
$$-2\left[\left\{\varphi\Theta_7'\cos\varphi\hat{t} - \varphi\beta_7'\sin\varphi\hat{t} - \frac{2r_3'}{3\varphi}\cos 2\varphi\hat{t} - \frac{3r_4'}{8\varphi}\cos 3\varphi\hat{t}\right\}\right.$$
$$\left.-\mu_2'\Theta_2'\varphi\cos\varphi\hat{t}\right] - \mu_2''\varphi\,\Theta_2\cos\varphi\hat{t} - 2\mu_2'\varphi\Theta_2\cos\varphi\hat{t}$$
$$-2\left[\beta_4'\cos\varphi\hat{t} + \tfrac{15\alpha}{4}\left\{\tfrac{r_0'}{\varphi^2} - \tfrac{B(\beta_1^2)'}{2\varphi^2}\cos 2\varphi\hat{t} - \tfrac{(\beta_1^3)'}{32\varphi^2}\cos 3\varphi\hat{t}\right\}\right] \qquad (6.63a)$$

To ensure a uniformly valid solution in $\hat{t}$ as far as (6.63a) is concerned, we equate to zero the coefficients of $\cos\varphi\hat{t}$ and $\sin\varphi\hat{t}$ and respectively get

$$\beta_7' + \beta_7 = -h_1(\tau) = -\tfrac{1}{2\varphi}\Theta_3\left(B^3 + \tfrac{\beta_1^2}{4}\right) \qquad (6.63b)$$

and

$$\Theta_7' + \Theta_7 = h_2(\tau)$$
$$= -\tfrac{1}{2\varphi}\left[\tfrac{45\alpha}{4}\left\{\tfrac{\beta_1\Theta_2^2}{4} + \beta_3\left(B^2 + \tfrac{3\beta_1^2}{4}\right)\right\} - 2\beta_4' - \beta_4'' - \mu_2''\varphi\Theta_2\right.$$
$$\left.-\mu_2'\varphi(2\Theta_2 + \Theta_2')\right] \qquad (6.63c)$$

On solving (6.63b,c), we get

$$\beta_7 = e^{-\tau}\left[\beta_7(0) - \int_0^\tau h_1(s)e^s ds\right]$$
$$\Theta_7 = e^{-\tau}\left[\Theta_7(0) + \int_0^\tau h_2(s)e^s ds\right]$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||
I.U. Udo-Akpan[1] *

The remaining equation in (6.60a) is

$$U_{m,\hat{t}\hat{t}}^{(32)} + \varphi^2 U_m^{(32)} = r_{11} + r_{14}\cos 2\varphi\hat{t} + r_{15}\sin 2\varphi\hat{t} + r_{16}\cos 3\varphi\hat{t} + r_{17}\sin 3\varphi\hat{t} \quad (6.64a)$$

$$U_m^{(32)}(0,0) = 0, \quad U_{m,\hat{t}}^{(32)}(0,0) + U_{m,\tau}^{(31)}(0,0) + \mu_2' U_{m,\hat{t}}^{(12)} = 0 \quad (6.64b)$$

where,

$$r_{11} = \frac{45\alpha}{4}\left(B\beta_1\beta_3 - \frac{B\theta_2^2}{2}\right) - \frac{15r_0'\alpha}{2\varphi^2} - \frac{15r_0''\alpha}{4\varphi^2} \quad (6.64c)$$

$$r_{14} = \frac{45\alpha}{4}\left(B\beta_1\beta_3 - \frac{B\theta_2^2}{2}\right) + \frac{15B(\beta_1^2)''}{8\varphi^2} + \frac{3(r_3 + r_3')}{3\varphi} + \frac{15\alpha B(\beta_1^2)'}{4\varphi^2} \quad (6.64d)$$

$$r_{15} = \frac{45\alpha B\beta_1\theta_2}{4} \quad (6.64e)$$

$$r_{16} = \frac{45\alpha}{16}(\beta_1^2\beta_3 - \theta_2^2\beta_1) + \frac{15\alpha(\beta_1^3)''}{128\varphi^2} + \frac{3(r_4' + r_4)}{4\varphi} \quad (6.64f)$$

$$r_{17} = \frac{45\alpha\theta_2\beta_1^2}{16} + \frac{15\alpha(\beta_1^3)'}{64\varphi^2} \quad (6.64g)$$

$$r_{11}(0) = \frac{-15\alpha B^3}{4\varphi^2}, \quad r_{14}(0) = \frac{195\alpha B^3}{12\varphi^2}, \quad r_{15}(0) = \frac{-45\alpha B^3}{4\varphi}, \quad (6.64h)$$

$$r_{16}(0) = \frac{-485\alpha B^3}{512\varphi^2}, \quad r_{17}(0) = \frac{-45\alpha B^3}{64\varphi^2} \quad (6.64i)$$

On solving (6.64a), using (6.64b) we get

$$U_m^{(32)}(\hat{t},\tau) = \beta_{10}\cos\varphi\hat{t} + \theta_{10}\sin\varphi\hat{t} + \frac{r_{11}}{\varphi^2} - \frac{1}{3\varphi^2}(r_{14}\cos 2\varphi\hat{t} + r_{15}\sin 2\varphi\hat{t})$$
$$- \frac{1}{8\varphi^2}(r_{16}\cos 3\varphi\hat{t} + r_{17}\sin 3\varphi\hat{t}) \quad (6.65a)$$

where,

$$\beta_{10}(0) = \frac{-r_{11}(0)}{\varphi^2} + \frac{r_{14}(0)}{3\varphi^2} + \frac{r_{16}(0)}{8\varphi^2}$$

$$\beta_{10}(0) = \frac{333555\alpha B^3}{36864}, \quad \theta_{10}(0) = 0 \quad (6.65b)$$

As was in the case of $U_m^{(32)}$ in (6.60a) which eventually led to (6.65a), we now substitute for terms into (6.61a) and to ensure a uniformly valid solution in $\hat{t}$, we equate to zero the coefficients of $\cos\varphi_{2m}\hat{t}$ and $\sin\varphi_{2m}\hat{t}$ and respectively get

$$\theta_8' + \theta_8 = \frac{-1}{2\varphi}(\beta_5'' + 2\beta_5') \quad (6.66a)$$

and

$$\beta_8' + \beta_8 = \frac{-1}{2\varphi}(\theta_5'' + 2\theta_5') \quad (6.66b)$$

On solving, we get

$$\theta_8 = e^{-\tau}\left[\theta_8(0) - \frac{1}{2\varphi}\int_0^\tau e^s(\beta_5'' + 2\beta_5')ds\right] \quad (6.66b)$$

$$\beta_8 = e^{-\tau}\left[\beta_8(0) - \frac{1}{2\varphi}\int_0^\tau e^s(\theta_5'' + 2\theta_5')ds\right] \quad (6.66c)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

The remaining equation in the substitution into (6.61a) is

$$U^{(32)}_{2m,\hat{t}\hat{t}} + \varphi^2_{2m}U^{(32)}_{2m} = r_{18} + r_{19}\cos\varphi\,\hat{t} + r_{20}\sin\varphi\,\hat{t} + r_{21}\cos 2\varphi\,\hat{t} + r_{22}\sin 2\varphi\,\hat{t} + r_{23}\cos 3\varphi\,\hat{t} + r_{23}\sin 3\varphi\,\hat{t} \qquad (6.67a)$$

$$U^{(32)}_{2m}(0,0) = 0, \qquad U^{(32)}_{2m,\hat{t}}(0,0) + U^{(31)}_{2m,\tau}(0,0) \qquad (6.67b)$$

where,

$$r_{18} = \frac{-9\alpha}{2}(B\theta_2^2 + B\beta_1\beta_3) + \frac{3r_0'\alpha}{U^2_{2m}} + \frac{3r_0''\alpha}{2U^2_{2m}} \qquad (6.67c)$$

$$r_{19} = \frac{-9\alpha}{2}\left\{\frac{\beta_1\theta_2^2}{4} + \beta_3\left(\frac{3\beta_1^2}{4} + B^2\right)\right\} + \frac{3\alpha r_1''}{2(U^2_{2m}-\varphi^2)} + \frac{3\alpha r_1'}{2(U^2_{2m}-\varphi^2)} - \frac{2\varphi(r_1'+r_1)}{U^2_{2m}-\varphi^2} \qquad (6.67d)$$

$$r_{20} = \frac{-9\alpha\theta_3}{2}\left(\frac{3\beta_1^2}{4} + B^2\right) \qquad (6.67e)$$

$$r_{21} = -9\alpha\left(\frac{B\theta_2^2}{2} + B\beta_1\beta_3\right) + \frac{9\alpha B(\beta_1^2)''}{2(U^2_{2m}-4\varphi^2)} + \frac{9\alpha B(\beta_1^3)'}{2(U^2_{2m}-4\varphi^2)} - \frac{4\varphi(r_6'+r_6)}{U^2_{2m}-4\varphi^2} \qquad (6.67f)$$

$$r_{22} = \frac{-9\alpha\theta_3\beta_1}{2} \qquad (6.67g)$$

$$r_{23} = -9\alpha\left(\frac{\theta_2^2\beta_1}{4} - \frac{\beta_1\beta_3}{4}\right) + \frac{9\alpha(\beta_1^3)''}{4(U^2_{2m}-9\varphi^2)} + \frac{3\alpha(\beta_1^3)'}{2(U^2_{2m}-9\varphi^2)} - \frac{6\varphi(r_7'+r_7)}{U^2_{2m}-9\varphi^2} \qquad (6.67h)$$

$$r_{24} = \frac{-9\alpha\theta_3\beta_1^2}{8} \qquad (6.67i)$$

On solving (6.67a,b), we get

$$U^{(32)}_{2m}(\hat{t},\tau) = \beta_{11}\cos\varphi_{2m}\hat{t} + \theta_{11}\sin\varphi_{2m}\hat{t} + \frac{r_{18}}{\varphi^2_{2m}} + \left(\frac{1}{\varphi^2_{2m} - \varphi^2}\right)(r_{19}\cos\varphi\hat{t} + r_{20}\sin\varphi\hat{t}) + \left(\frac{1}{\varphi^2_{2m} - 4\varphi^2}\right)(r_{21}\cos 2\varphi\hat{t} + r_{22}\sin 2\varphi\hat{t}) + \left(\frac{1}{\varphi^2_{2m} - 9\varphi^2}\right)(r_{23}\cos 3\varphi\hat{t} + r_{24}\sin 3\varphi\hat{t}) \qquad (6.68a)$$

We may not need $\beta_{11}(0)$ and $\theta_{11}(0)$. We next substitute into (6.62a) and to ensure a uniformly valid solution in $\hat{t}$, equate to zero, the coefficients of $\cos\varphi_{3m}\hat{t}$ and $\sin\varphi_{3m}\hat{t}$ and respectively get

$$\theta_9' + \theta_9 = \frac{-1}{2\varphi_{3m}}(\beta_6'' + 2\beta_6') \qquad (6.68b)$$

and

$$\beta_9' + \beta_9 = \frac{1}{2\varphi_{3m}}(\theta_6'' + 2\theta_6') \qquad (6.68c)$$

On solving, we get

$$\theta_9 = e^{-\tau}\left[\theta_9(0) - \frac{1}{2\varphi_{3m}}\int_0^\tau e^s(\beta_6'' + 2\beta_6')ds\right] \qquad (6.68d)$$

$$\beta_9 = e^{-\tau}\left[\beta_9(0) + \frac{-1}{2\varphi_{3m}}\int_0^\tau e^s(\theta_6'' + 2\theta_6')ds\right] \qquad (6.68e)$$

The remaining equation in the substitution into (6.62a) is

$$U^{(32)}_{3m,\hat{t}\hat{t}} + \varphi^2_{3m}U^{(32)}_{3m} = r_{25} + r_{26}\cos\varphi\,\hat{t} + r_{27}\sin\varphi\,\hat{t} + r_{28}\cos 2\varphi\,\hat{t} + r_{29}\sin 2\varphi\,\hat{t} + r_{30}\cos 3\varphi\,\hat{t} + r_{31}\sin 3\varphi\,\hat{t} \qquad (6.69a)$$

$$U^{(32)}_{3m}(0,0) = 0, \quad U^{(32)}_{3m,\hat{t}}(0,0) + U^{(31)}_{3m,\tau}(0,0) \qquad (6.69b)$$

where,

$$r_{25} = \frac{3\alpha B\theta_2^2}{8} + \frac{3B\beta_1\beta_3\alpha}{4} - \alpha\frac{(r_0'' + r_0')}{4U_{3m}^2} \quad (6.69c)$$

$$r_{26} = \frac{3\alpha\theta_2^2\beta_1}{16} + \frac{3\alpha\beta_3}{4}\left(\frac{3\beta_1^2}{4} + B^2\right) - \frac{\alpha r_1''}{4(U_{3m}^2-\varphi^2)} - \frac{2\varphi(r_8'+r_0)}{U_{3m}^2-\varphi^2} - \frac{-\alpha r_1'}{2(U_{3m}^2-\varphi^2)} \quad (6.69d)$$

$$r_{27} = 3\alpha\theta_3\left(B^2 - \frac{\beta_1^2}{4}\right) \quad (6.69e)$$

$$r_{28} = \frac{-3\alpha B\theta_2^2}{8} + \frac{3B\beta_1\beta_3\alpha}{4} - \frac{3\alpha B(\beta_1^2)''}{8(U_{3m}^2-4\varphi^2)} - \frac{3\alpha B(\beta_1^2)'}{U_{2m}^2-4\varphi^2} - \frac{4\varphi(r_9'+r_9)}{U_{3m}^2-4\varphi^2} \quad (6.69f)$$

$$r_{29} = \frac{3\alpha\theta_2 B\beta_1}{4} \quad (6.69g)$$

$$r_{30} = -\frac{-3\alpha\beta_1\theta_2^2}{8} + \frac{3\beta_1^2\beta_3\alpha}{16} - \frac{\alpha(\beta_1^3)''}{4(U_{3m}^2-9\varphi^2)} - \frac{\alpha(\beta_1^3)'}{2(U_{3m}^2-9\varphi^2)} - \frac{6\varphi(r_{10}'+r_{10})}{U_{3m}^2-9\varphi^2} \quad (6.69h)$$

$$r_{31} = \frac{3\alpha\theta_3\beta_1^2}{16} \quad (6.69i)$$

$$r_{25}(0) = \frac{3\alpha B^3}{8\varphi^2}, \quad r_{26}(0) = 3\alpha B^3 F_{14} \quad (6.69j)$$

$$F_{14} = \frac{-3}{16\varphi^2} + \frac{39}{16(U_{3m}^2-\varphi^2)} + \frac{2\alpha\varphi}{U_{3m}^2-\varphi^2}\left(\frac{57}{128\varphi} + \frac{9\varphi}{4(U_{3m}^2-9\varphi^2)}\right) - \frac{21}{8(U_{3m}^2-\varphi^2)} \quad (6.69k)$$

$$r_{27}(0) = 0, \quad r_{28}(0) = 3\alpha B^3 F_{15} \quad (6.69l)$$

$$F_{15} = \frac{-3}{8\varphi^2} - \frac{3}{2(U_{3m}^2-\varphi^2)} - \frac{4\varphi}{U_{3m}^2-\varphi^2}\left(\frac{57}{32\varphi} - \frac{3\varphi}{U_{3m}^2-\varphi^2}\right) + \frac{6}{U_{3m}^2-\varphi^2} \quad (6.69m)$$

$$r_{29}(0) = 0, \quad r_{30}(0) = 3\alpha B^3 F_{16}, \quad r_{31}(0) = 0 \quad (6.69n)$$

$$F_{16} = \frac{3}{16\varphi^2} + \frac{61}{56(U_{3m}^2-9\varphi^2)} \quad (6.69o)$$

On solving (6.69a,b), we get

$$U_{3m}^{32}(\hat{t},\tau) = \beta_{12}\cos\varphi_{3m}\hat{t} + \theta_{12}\sin\varphi_{3m}\hat{t} + \frac{r_{25}}{\varphi_{3m}^2} + \left(\frac{1}{\varphi_{3m}^2 - \varphi^2}\right)(r_{26}\cos\varphi\hat{t} + r_{27}\sin\varphi\hat{t})$$
$$+ \left(\frac{1}{\varphi_{3m}^2 - 4\varphi^2}\right)(r_{28}\cos 2\varphi\hat{t} + r_{29}\sin 2\varphi\hat{t})$$
$$+ \left(\frac{1}{\varphi_{3m}^2 - 9\varphi^2}\right)(r_{30}\cos 3\varphi\hat{t} + r_{31}\sin 3\varphi\hat{t}) \quad (6.70a)$$

where,

$$\beta_{12}(0) = 3\alpha B^3 F_{17}, \quad (6.70b)$$

$$F_{17} = -\left[\frac{1}{8\varphi^2\varphi_{3m}^2} + \frac{F_{14}}{\varphi_{3m}^2 - \varphi^2} + \frac{F_{15}}{\varphi_{3m}^2 - 4\varphi^2} + \frac{F_{16}}{\varphi_{3m}^2 - 9\varphi^2}\right] \quad (6.70c)$$

$$\theta_{12}(0) = 0 \quad (6.70d)$$

So far, we write the deflection as

$$w(x,\hat{t},\tau) = \epsilon\left(U_m^{(10)} + \delta U_m^{(11)} + \delta^2 U_m^{(12)} + \cdots\right)(1 - \cos 2mx)$$
$$+\epsilon^3\Big[U_m^{(30)}(1-\cos 2mx) + U_{2m}^{(30)}(1-\cos 4mx) + U_{3m}^{(30)}(1-\cos 6mx)$$
$$+\delta\left\{U_m^{(31)}(1-\cos 2mx) + U_{2m}^{(31)}(1-\cos 4mx) + U_{3m}^{(31)}(1-\cos 6mx)\right\}$$
$$+\delta^2\left\{U_m^{(32)}(1-\cos 2mx) + U_{2m}^{(32)}(1-\cos 4mx) + U_{3m}^{(32)}(1-\cos 6mx)\right\}$$
$$+\cdots\Big] + \cdots \quad (6.71)$$

7. CRITICAL VALUES OF THE DEPENDENT VARIABLES AT MAXIMUM DEFLECTION

As in [18, 19], the dynamic buckling load λ_D, which is defined as the largest load parameter for the solution of the problem to be bounded, is obtained from the maximization

$$\frac{d\lambda}{dw_a} = 0 \qquad (7.1)$$

where, w_a is the maximum deflection whose maximum values of the dependent variables we shall now determine.

Let $t_a, \hat{t}_a, \tau_a$ be the values, at maximum displacement of $t, \hat{t}$and τ respectively and let us now assume the following asymptotic series

$$\hat{t}_a = \hat{t}_0 + \delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots) + \cdots \qquad (7.2a)$$

$$t_a = t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots) + \cdots$$

$$\tau_a = \delta t_a = \delta[t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots) + \cdots] \qquad (7.2b)$$

As a function of $x, \hat{t}, \tau$, the conditions for $w(x, \hat{t}, \tau)$ to have a maximum are

$$w_{,x} = 0, \quad (1 + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)w_{,\hat{t}} + \delta w_{,\tau} = 0 \qquad (7.3)$$

On substituting (6.71) into the first of (7.3), we get the value of x at maximum deflection, namely x_a, as

$$x_a = \frac{\pi}{2m}, \quad m = 1, 2, 3, \ldots \qquad (7.4)$$

On evaluating (6.71) at $x = x_a = \frac{\pi}{2m}$, we get

$$w = 2\epsilon\left(U_m^{(10)} + \delta U_m^{(11)} + \delta^2 U_m^{(12)} + \cdots\right) + 2\epsilon^3\left[U_m^{(30)} + U_{3m}^{(30)} + \delta\left\{U_m^{(31)} + U_{3m}^{(31)}\right\}\right.$$
$$\left. + \delta^2\left\{U_m^{(32)} + U_{3m}^{(32)}\right\} + \cdots\right] \qquad (7.5)$$

We shall now expand each of the terms of (7.3), (which is evaluated at $(x_a, \hat{t}_a, \tau_a)$) by using (7.2a-c), as well as (7.4) and (7.5). Thus, we get

$$\epsilon U_{m,\hat{t}}^{(10)} = \epsilon\left[U_{m,\hat{t}}^{(10)} + \{\delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots) + \cdots\}U_{m,\hat{t}\hat{t}}^{(10)}\right.$$
$$+ \delta\{t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots) + \cdots\}U_{m,\hat{t}\tau}^{(10)}$$
$$+ \frac{1}{2}\left\{\{\delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots) + \cdots\}^2 U_{m,\hat{t}\hat{t}\hat{t}}^{(10)}\right.$$
$$+ 2\delta\{\delta\hat{t}_{01} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots) + \cdots\}$$
$$\times\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}U_{m,\hat{t}\hat{t}\tau}^{(10)}$$
$$\left.\left. + \delta^2\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots) + \cdots\}U_{m,\hat{t}\tau\tau}^{(10)}\right\}\right]\Bigg|_{(\hat{t}_0, 0)} \qquad (7.6a)$$

$$\epsilon\delta U_{m,\hat{t}}^{(11)} = \epsilon\delta\left[U_{m,\hat{t}}^{(11)} + \{\delta\hat{t}_{01} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \ldots)\}U_{m,\hat{t}\hat{t}}^{(11)}\right.$$
$$+ \delta\{t_0 + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots) + \cdots\}U_{m,\hat{t}\tau}^{(11)}$$
$$+ \frac{1}{2}\left\{\{\delta\hat{t}_{01} + \ldots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots) + \cdots\}^2 U_{m,\hat{t}\hat{t}\hat{t}}^{(11)}\right.$$
$$+ 2\delta\{\delta\hat{t}_{01} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots) + \cdots\}$$
$$\times\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}U_{m,\hat{t}\hat{t}\tau}^{(11)}$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

$$+\delta^2\{t_0+\delta t_{01}+\cdots+\epsilon^2(t_{20}+\delta t_{21}+\cdots)+\cdots\}U^{(11)}_{m,\hat{t}\tau\tau}\}]\Big|_{(\hat{t}_0,0)} \quad (7.6b)$$

$$\epsilon\delta^2U^{(12)}_{m,\hat{t}} = \epsilon\delta^2\Big[U^{(12)}_{m,\hat{t}}+\{\delta\hat{t}_{01}+\cdots+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\ldots)\}U^{(12)}_{m,\hat{t}\hat{t}}$$
$$+\delta\{t_0+\cdots+\epsilon^2(t_{20}+\delta t_{21}+\cdots)+\cdots\}U^{(12)}_{m,\hat{t}\tau}+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6c)$$

$$\epsilon^3\left(U^{(30)}_{m,\hat{t}}+U^{(30)}_{3m,\hat{t}}\right)=\epsilon^3\Big[\left(U^{(30)}_{m,\hat{t}}+U^{(30)}_{3m,\hat{t}}\right)+\{\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots\}\left(U^{(30)}_{m,\hat{t}\hat{t}}+U^{(30)}_{3m,\hat{t}\hat{t}}\right)$$
$$+\delta\{t_0+\delta t_{01}+\cdots\}\left(U^{(30)}_{m,\hat{t}\tau}+U^{(30)}_{3m,\hat{t}\tau}\right)+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6d)$$

$$\epsilon^3\delta\left(U^{(31)}_{m,\hat{t}}+U^{(31)}_{3m,\hat{t}}\right)=\epsilon^3\delta\Big[\left(U^{(31)}_{m,\hat{t}}+U^{(31)}_{3m,\hat{t}}\right)+\{\delta\hat{t}_{01}+\cdots\}\left(U^{(31)}_{m,\hat{t}\hat{t}}+U^{(31)}_{3m,\hat{t}\hat{t}}\right)$$
$$+\delta\{t_0+\delta t_{01}+\cdots\}\left(U^{(31)}_{m,\hat{t}\tau}+U^{(31)}_{3m,\hat{t}\tau}\right)+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6e)$$

$$\epsilon^3\delta^2\left(U^{(32)}_{m,\hat{t}}+U^{(32)}_{3m,\hat{t}}\right)=\epsilon^3\delta^2\Big[\left(U^{(32)}_{m,\hat{t}}+U^{(32)}_{3m,\hat{t}}\right)+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6f)$$

$$\epsilon^3\mu_2'U^{(10)}_{m,\hat{t}}=\epsilon^3\Big[\mu_2'U^{(10)}_{m,\hat{t}}+\mu_2'\{\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots\}U^{(10)}_{m,\hat{t}\hat{t}}+\delta\{t_0+\delta t_{01}+\cdots\}\left(\mu_2'U^{(10)}_{m,\hat{t}}\right)_{,\tau}$$
$$+\frac{1}{2}\{\hat{t}_{01}+\cdots\}^2\mu_2'U^{(10)}_{m,\hat{t}\hat{t}}+\delta\{t_0+\delta t_{01}+\cdots\}\{\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots\}\left(\mu_2'U^{(10)}_{m,\hat{t}\hat{t}}\right)_{,\tau}$$
$$+\frac{1}{2}\delta^2(t_0+\cdots)^2\left(\mu_2'U^{(10)}_{m,\hat{t}}\right)_{,\tau\tau}+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6g)$$

$$\epsilon^3\delta\mu_2'U^{(11)}_{m,\hat{t}}=\epsilon^3\delta\Big[\mu_2'U^{(11)}_{m,\hat{t}}+(\delta\hat{t}_{01}+\cdots)\mu_2'U^{(11)}_{m,\hat{t}\hat{t}}$$
$$+\delta(t_0+\cdots)\left(\mu_2'U^{(11)}_{m,\hat{t}\hat{t}}\right)_{,\tau}+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6h)$$

$$\epsilon^3\delta^2\mu_2'U^{(12)}_{m,\hat{t}}=\epsilon^3\delta^2\Big[\mu_2'U^{(12)}_{m,\hat{t}}+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6i)$$

$$\epsilon\delta U^{(10)}_{m,\tau}=\epsilon\delta\Big[U^{(10)}_{m,\tau}+\{\delta\hat{t}_{01}+\cdots+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\ldots)\}U^{(10)}_{m,\hat{t}\tau}$$
$$+\frac{1}{2}\{\{\delta\hat{t}_{01}+\ldots+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)+\cdots\}^2U^{(11)}_{m,\hat{t}\hat{t}\tau}$$
$$+2\delta\{\delta\hat{t}_{01}+\cdots+\epsilon^2(\hat{t}_{20}+\ldots)+\cdots\}$$
$$\times\{t_0+\delta t_{01}+\cdots+\epsilon^2(t_{20}+\delta t_{21}+\cdots)\}U^{(11)}_{m,\hat{t}\tau\tau}+\cdots\}\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6j)$$

$$\epsilon\delta^2U^{(11)}_{m,\tau}=\epsilon\delta^2\Big[U^{(11)}_{m,\tau}+\cdots+\{\delta\hat{t}_{01}+\cdots+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}U^{(11)}_{m,\hat{t}\tau}+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6k)$$

$$\epsilon^3\delta\left(U^{(30)}_{m,\tau}+U^{(30)}_{3m,\tau}\right)=\epsilon^3\delta\Big[\left(U^{(30)}_{m,\tau}+U^{(30)}_{3m,\tau}\right)+\left\{(\delta\hat{t}_{01}+\ldots)\left(U^{(30)}_{m,\hat{t}\tau}+U^{(30)}_{3m,\hat{t}\tau}\right)+\cdots\right\}$$
$$+\delta(t_0+\ldots)\left(U^{(30)}_{m,\tau\tau}+U^{(30)}_{3m,\tau\tau}\right)+\cdots\Big]\Big|_{(\hat{t}_0,0)} \quad (7.6l)$$

$$\epsilon^3\delta^2\left(U^{(31)}_{m,\tau}+U^{(31)}_{3m,\tau}\right)=\epsilon^3\delta\left(U^{(31)}_{m,\tau}+U^{(31)}_{3m,\tau}\right)\Big|_{(\hat{t}_0,0)}+\cdots \quad (7.6m)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

By substituting (7.6a – m) into the second equation of (7.3) and equating the coefficients of $\epsilon^i\delta^j$, we get

$$O(\epsilon):\quad U_{m,\hat{t}}^{(10)}(\hat{t}_0,0)=0 \tag{7.7a}$$

$$O(\epsilon\delta):\quad \hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(10)}+t_0U_{m,\hat{t}\tau}^{(10)}+U_{m,\tau}^{(10)}=0 \tag{7.7b}$$

$$O(\epsilon\delta^2):\quad \hat{t}_{02}U_{m,\hat{t}\hat{t}}^{(10)}+t_{01}U_{m,\hat{t}\tau}^{(10)}+\frac{(\hat{t}_{01})^2}{2}U_{m,\hat{t}\hat{t}}^{(10)}+\hat{t}_{01}t_0U_{m,\hat{t}\hat{t}\tau}^{(10)}+\frac{(t_0)^2}{2}U_{m,\hat{t}\tau\tau}^{(10)}$$
$$+\hat{t}_{01}U_{m,\hat{t}\hat{t}}^{(11)}+t_0U_{m,\hat{t}\tau}^{(10)}+U_{m,\hat{t}}^{(12)}+\hat{t}_{01}U_{m,\hat{t}\tau}^{(10)}+t_0U_{m,\tau\tau}^{(10)}+U_{m,\tau}^{(11)} \tag{7.7c}$$

$$O(\epsilon^3):\quad \hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(10)}+\left(U_{m,\hat{t}\hat{t}}^{(30)}+U_{3m,\hat{t}\hat{t}}^{(30)}\right)+\mu_2'(0)U_{m,\hat{t}}^{(10)}=0 \tag{7.7d}$$

$$O(\epsilon^3\delta):\quad \hat{t}_{21}U_{m,\hat{t}\hat{t}}^{(10)}+t_{20}U_{m,\hat{t}\tau}^{(10)}+\frac{1}{2}\left\{2\hat{t}_{01}\hat{t}_{20}U_{m,\hat{t}\hat{t}\hat{t}}^{(10)}+2\hat{t}_{20}t_0U_{m,\hat{t}\hat{t}\tau}^{(10)}\right\}+\hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(11)}$$
$$+\hat{t}_{01}\left(U_{m,\hat{t}\hat{t}}^{(30)}+U_{3m,\hat{t}\hat{t}}^{(30)}\right)+t_0\left(U_{m,\hat{t}\tau}^{(30)}+U_{3m,\hat{t}\tau}^{(30)}\right)+\left(U_{m,\hat{t}}^{(30)}+U_{3m,\hat{t}}^{(30)}\right)$$
$$+\hat{t}_{20}U_{m,\hat{t}\tau}^{(11)}+\left(U_{m,\tau}^{(30)}+U_{3m,\tau}^{(30)}\right)=0 \tag{7.7e}$$

etc., where equations (7.7a) to (7.7e) are evaluated at $(\hat{t}_0,0)$.

From (7.7a), we get

$$\varphi\hat{t}_0=\pi n,\qquad n=1,2,3,\dots$$
$$\therefore\qquad \hat{t}_0=\frac{\pi}{\varphi},\qquad (n=1) \tag{7.8a}$$

where we have taken $n=1$.

On substituting (7.8) in (7.7b) and simplifying, we get

$$\hat{t}_{01}=\left.\frac{-\left(U_{m,\hat{t}}^{(11)}+U_{m,\tau}^{(10)}+U_{m,\hat{t}\tau}^{(10)}\right)}{U_{m,\hat{t}\hat{t}}^{(10)}}\right|_{(\hat{t}_0,0)}=0 \tag{7.8b}$$

On substituting for terms in (7.7c) and simplifying, we get

$$\hat{t}_{02}=\left.\frac{-\hat{t}_0U_{m,\tau\tau}^{(10)}}{U_{m,\hat{t}\hat{t}}^{(10)}}\right|_{(\hat{t}_0,0)}=\frac{\hat{t}_0}{\varphi^2}=\frac{\pi}{\varphi^3} \tag{7.8c}$$

We next substitute into (7.7d) and get

$$\hat{t}_{20}=\left.\frac{-\left(U_{m,\hat{t}\hat{t}}^{(30)}+U_{3m,\hat{t}\hat{t}}^{(30)}\right)}{U_{m,\hat{t}\hat{t}}^{(10)}}\right|_{(\hat{t}_0,0)} \tag{7.8d}$$

Now,

$$U_{m,\hat{t}\hat{t}}^{(30)}(\hat{t}_0,0)=\frac{45\alpha B^3}{16},\qquad U_{3m,\hat{t}\hat{t}}^{(30)}(\hat{t}_0,0)=\alpha B^3S_{15}, \tag{7.8e}$$

$$S_{15}=\frac{1}{4}\left[\left\{\frac{1}{2\varphi_{3m}^2}-\frac{15}{2(\varphi_{3m}^2-\varphi^2)}+\frac{3}{2(\varphi_{3m}^2-4\varphi^2)}-\frac{1}{2(\varphi_{3m}^2-9\varphi^2)}\right\}\right.$$
$$\left.-\left\{\frac{15\varphi^2}{4(\varphi_{3m}^2-\varphi^2)}+\frac{12\varphi^2}{(\varphi_{3m}^2-4\varphi^2)}-\frac{9\varphi^2}{2(\varphi_{3m}^2-9\varphi^2)}\right\}\right] \tag{7.8f}$$

$$U_{m,\hat{t}\hat{t}}^{(10)}(\hat{t}_0,0)=-\varphi^2B \tag{7.8g}$$

On substituting for terms in (7.8d), we get

$$\hat{t}_{20}=\alpha B^2T_{20},\qquad T_{20}=\left(S_{15}+\frac{45}{16}\right) \tag{7.8h}$$

After substituting for terms in (7.7e) and simplifying, many terms vanish and the remaining terms give

$$\hat{t}_{21} = \left[\frac{-1}{U^{(10)}_{m,\hat{t}\hat{t}}}\left(\hat{t}_{20}t_0 + U^{(10)}_{m,\hat{t}\hat{t}\tau} + U^{(31)}_{m,\hat{t}} + U^{(31)}_{3m,\hat{t}} + U^{(30)}_{m,\tau} + U^{(30)}_{3m,\tau}\right)\right]\Bigg|_{(\hat{t}_0,0)} \qquad (7.8i)$$

We now evaluate each of the terms in (7.8i) as follows:

$$U^{(31)}_{m,\hat{t}}(\hat{t}_0,0) = \frac{2385\alpha B^3}{256} \qquad (7.9a)$$

$$U^{(31)}_{3m,\hat{t}}(\hat{t}_0,0) = \alpha B^3 F_{20} \qquad (7.9b)$$

$$F_{20} = S_1 \cos\varphi_{3m}\hat{t}_0 - \frac{\varphi F_1}{2(\varphi_{3m}^2 - \varphi^2)} - \frac{3}{28(\varphi_{3m}^2 - 4\varphi^2)} - \frac{9}{56(\varphi_{3m}^2 - 9\varphi^2)} \qquad (7.9c)$$

$$U^{(30)}_{m,\tau}(\hat{t}_0,0) = \frac{315\alpha B^3}{64\varphi^2} \qquad (7.9d)$$

$$U^{(30)}_{3m,\tau}(\hat{t}_0,0) = \alpha B^3 S_{47} \qquad (7.9e)$$

$$S_{47} = -\frac{F_4\cos\varphi_{3m}\hat{t}_0}{8} - \frac{1}{4}\left[\frac{3}{\varphi_{3m}^2} + \frac{21}{4(\varphi_{3m}^2 - \varphi^2)} + \frac{3}{28(\varphi_{3m}^2 - 4\varphi^2)} + \frac{3}{56(\varphi_{3m}^2 - 9\varphi^2)}\right] \qquad (7.9f)$$

$$U^{(10)}_{m,\hat{t}\hat{t}\tau}(\hat{t}_0,0) = \varphi^2\beta_1'(0) = \varphi^2 B \qquad (7.9g)$$

On substituting for terms in (7.8i), we get

$$\hat{t}_{21} = \alpha B^2 T_{21}, \qquad T_{21} = \frac{1}{\varphi^2}\left[t_0 T_{20}\varphi^2 + \frac{2385}{256} + F_{20} - \frac{13779}{384\varphi^2} + S_{47}\right] \qquad (7.10)$$

Later, we shall also need terms like $t_0, t_{01}, t_{02}, t_{20}$ *and* t_{21} which we now evaluate directly from (6.2b) (evaluated at maximum values of the variables). Thus, at maximum deflection, (6.2b) becomes

$$\hat{t}_a = t_a + \left(\frac{\mu_2(\tau_a)\epsilon^2 + \mu_3(\tau_a)\epsilon^3 + \cdots}{\delta}\right) \qquad (7.11)$$

By using (7.2a – c), we can write (7.11) as

$$\begin{aligned}\hat{t}_0 + \delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots) + \cdots \\ = t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots) \\ + \epsilon^2\left\{\mu_2(0) + t_a\mu_2'(0) + \frac{\delta t_a^2}{2}\mu_2''(0)\right\} \\ + \epsilon^3\left\{\mu_3(0) + t_a\mu_3'(0) + \frac{\delta t_a^2}{2}\mu_3''(0) + \cdots\right\} + \cdots\end{aligned} \qquad (7.12)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

We note the fact that $\mu_i(0) = 0$ and equally note that t_a can be expanded using (7.2b). By equating coefficients of $\epsilon^i \delta^j$ in (7.12), we get

$$O(\epsilon): \qquad \hat{t}_0 = \frac{\pi}{\varphi} = t_0$$

$$\therefore \qquad t_0 = \frac{\pi}{\varphi} \qquad (7.13a)$$

$$O(\epsilon\delta): \quad \hat{t}_{01} = 0 = t_{01}$$

$$\therefore \qquad t_{01} = 0 \qquad (7.13b)$$

$$O(\epsilon\delta^2): \qquad \hat{t}_{02} = \frac{t_0}{\varphi} = \frac{\pi}{\varphi^2} = t_{02}$$

$$\therefore \qquad t_{02} = \frac{\pi}{\varphi^2} \qquad (7.13c)$$

$$O(\epsilon^3): \qquad \hat{t}_{20} = \alpha B^2 T_{20} = t_{20} + \mu_2'(0) t_0$$

$$\therefore \qquad t_{20} = \alpha B^2 T_{20} - \mu_2'(0) t_0 = \alpha B^2 T_{20}^{(1)} \qquad (7.13d)$$

$$T_{20}^{(1)} = T_{20} + \frac{225\pi}{32\varphi^3} \qquad (7.14)$$

$$O(\epsilon^3\delta): \quad \hat{t}_{21} = \alpha B^2 T_{21} = t_{21} + t_{01}\mu_2'(0) + \frac{t_0^2}{2}\mu_2''(0)$$

$$\therefore t_{21} = \alpha B^2 T_{21} - t_{01}\mu_2'(0) - \frac{t_0^2}{2}\mu_2''(0) = \alpha B^2 T_{21}^{(1)} \qquad (7.15a)$$

$$T_{21}^{(1)} = T_{21} - \frac{45\pi^2}{32\varphi^4} \qquad (7.15b)$$

8. MAXIMUM DEFLECTION, w_a

To determine the maximum deflection $\mathrm{w_a}$, we evaluate (7.5) at $\hat{t}_a$, t_a *and* τ_a. Thus, we get

$$\mathrm{w_a} = 2\epsilon\left(U_{ma}^{(10)} + \delta U_{ma}^{(11)} + \delta^2 U_{ma}^{(12)}\right) + 2\epsilon^3\left[\left(U_{ma}^{(30)} + U_{3ma}^{(30)}\right) + \delta\left(U_{ma}^{(31)} + U_{3ma}^{(31)}\right) + \delta^2\left(U_{ma}^{(32)} + U_{3ma}^{(32)}\right) + \cdots\right] + \cdots \qquad (8.1)$$

where, $U_{ma}^{(ij)} = U_m^{(ij)}(\hat{t}_a, \tau_a)$.

We now expand each term of (8.1) using (7.2a – c). Thus, we have

$$\epsilon U_{ma}^{(10)} = \epsilon U_m^{(10)}(\hat{t}_a, t_a, \tau_a) = \epsilon\Big[U_m^{(10)} + \{\hat{t}_{01}\delta + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots)\}$$
$$+\frac{1}{2}\{\{\hat{t}_{01}\delta + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots)\}^2$$
$$+ \delta\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}$$
$$\times\{\hat{t}_{01}\delta + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots)\}U_{m,\hat{t}}^{(10)}$$
$$+\delta^2\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}^2 + \cdots\Big]\Big|_{(\hat{t}_0, 0)} \qquad (8.2)$$

$$\epsilon\delta U_{ma}^{(11)} = \epsilon\delta\Big[U_m^{(11)} + \{\hat{t}_{01}\delta + \delta^2\hat{t}_{02} + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots)\}U_{m,\hat{t}}^{(11)}$$
$$+ \delta\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}$$
$$+\frac{1}{2}\{\{\hat{t}_{01}\delta + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots)\}^2$$
$$+ 2\delta\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}$$
$$\times\{\hat{t}_{01}\delta + \cdots + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots)\}U_{m,\hat{t}\tau}^{(11)}$$

$$+\delta^2\{t_0 + \delta t_{01} + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}^2 + \cdots]|_{(\hat{t}_0,0)} \quad (8.3)$$

$$\epsilon\delta^2 U_{ma}^{(12)} = \epsilon\delta^2 \Big[U_m^{(12)} + \{\hat{t}_{01}\delta + \cdots + \epsilon^2(\hat{t}_{20} + \cdots)\}U_{m,\hat{t}}^{(11)} + \delta\{t_0 + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\} + \cdots]|_{(\hat{t}_0,0)} \quad (8.4)$$

$$\epsilon^3\left(U_m^{(30)} + U_{3m}^{(30)}\right) = \epsilon^3\Big[\left(U_m^{(30)} + U_{3m}^{(30)}\right) + \{\hat{t}_{01}\delta + \cdots\}\left(U_m^{(30)} + U_{3m}^{(30)}\right)_{,\hat{t}} + \delta\{t_0 + \cdots + \epsilon^2(t_{20} + \delta t_{21} + \cdots)\}\left(U_{m,\tau}^{(30)} + U_{3m,\tau}^{(30)}\right)\Big]\Big|_{(\hat{t}_0,0)} \quad (8.5)$$

$$\delta\epsilon^3\left(U_m^{(31)} + U_{3m}^{(31)}\right) = \delta\epsilon^3\Big[\left(U_m^{(31)} + U_{3m}^{(31)}\right) + \{\hat{t}_{01}\delta + \cdots\}\left(U_m^{(31)} + U_{3m}^{(31)}\right)_{,\hat{t}} + \delta(t_0 + \cdots)\left(U_{m,\tau}^{(31)} + U_{3m,\tau}^{(31)}\right)\Big]\Big|_{(\hat{t}_0,0)} \quad (8.6)$$

$$\delta^2\epsilon^3\left(U_m^{(32)} + U_{3m}^{(32)}\right) = \delta^2\epsilon^3\left(U_{m,\tau}^{(31)} + U_{3m,\tau}^{(31)}\right)\Big]\Big|_{(\hat{t}_0,0)} \quad (8.7)$$

On substituting (8.2) – (8.7) into (8.1), we observe that most terms vanish and the remaining ones are given as

$$\mathrm{w_a} = 2\epsilon\left(U_m^{(10)} + \delta t_0 U_{m,\tau}^{(10)} + \frac{\delta^2 t_0^2}{2}U_{m,\tau\tau}^{(10)}\right)\Big]\Big|_{(\hat{t}_0,0)}$$
$$+2\epsilon^3\Big[\left(U_m^{(30)} + U_{3m}^{(30)}\right) + \delta\left\{t_{20}U_{m,\tau}^{(10)} + \hat{t}_{20}U_{m,\hat{t}}^{(10)} + t_0\left(U_{m,\tau}^{(30)} + U_{3m,\tau}^{(30)}\right)\right\}$$
$$+\delta^2\Big\{t_{21}U_{m,\tau}^{(10)} + \hat{t}_{02}\hat{t}_{20}U_{m,\hat{t}\hat{t}}^{(10)} + t_0 t_{20}U_{m,\tau\tau}^{(10)} + \hat{t}_{21}U_{m,\hat{t}}^{(11)} + t_0\hat{t}_{20}U_{m,\hat{t}\tau}^{(11)}$$
$$+\frac{t_0^2}{2}\left(U_{m,\tau}^{(30)} + U_{3m,\tau}^{(30)}\right) + \left(U_m^{(32)} + U_{3m}^{(32)}\right)\Big\}\Big]\Big|_{(\hat{t}_0,0)} + \cdots \quad (8.8)$$

We however note that

$$U_m^{(30)}(\hat{t}_0,0) = \frac{15\alpha B^3}{\varphi^2}, \quad (8.9a)$$

$$U_{3m}^{(30)}(\hat{t}_0,0) = \alpha B^3 S_9 \quad (8.9b)$$

$$S_9 = \frac{1}{4}\left[\frac{5(1-\cos\varphi_{3m}\hat{t}_0)}{2\varphi^2} + \frac{5(1+\cos\varphi_{3m}\hat{t}_0)}{4(\varphi_{3m}^2-\varphi^2)} + \frac{3(1-\cos\varphi_{3m}\hat{t}_0)}{2(\varphi_{3m}^2-4\varphi^2)} + \frac{(1-\cos\varphi_{3m}\hat{t}_0)}{4(\varphi_{3m}^2-9\varphi^2)}\right] \quad (8.9c)$$

$$U_m^{(32)}(\hat{t}_0,0) = \frac{-55\alpha B^3}{3\varphi^4}, \quad (8.10a)$$

$$U_{3m}^{(30)}(\hat{t}_0,0) = \alpha B^3 S_{17} \quad (8.10b)$$

$$S_{17} = \left[\frac{3(1-\cos\varphi_{3m}\hat{t}_0)}{8\varphi_{3m}^2\varphi^2} - \frac{F_{14}(1+\cos\varphi_{3m}\hat{t}_0)}{\varphi_{3m}^2-\varphi^2} + \frac{F_{15}(1-\cos\varphi_{3m}\hat{t}_0)}{\varphi_{3m}^2-4\varphi^2} - \frac{F_{16}(1+\cos\varphi_{3m}\hat{t}_0)}{4(\varphi_{3m}^2-9\varphi^2)}\right] \quad (8.10c)$$

On substituting (8.9a) – (8.10c) into (8.8) and simplifying, we get

$$\mathrm{w_a} = 4\epsilon B\left[1 - \frac{\delta\pi}{2\varphi} + \frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2\right] + \frac{30\alpha B^3\epsilon^3}{\varphi^2}\left[\left(1 + \frac{\varphi^2 S_9}{15}\right) + A_{31}\delta + A_{32}\delta^2\right] + \cdots \quad (8.11a)$$

where,

$$A_{31} = \frac{\varphi^2}{15}\left[\frac{\pi}{\varphi}\left(\frac{315}{64\varphi^2} + S_{47}\right) - T_{20}^{(1)} - T_{20}\right] \quad (8.11b)$$

$$A_{32} = \frac{\varphi^2}{15}\left[\left(\frac{\pi}{\varphi}\right)T_{20}^{(1)} - T_{21}^{(1)} - \left(\frac{\pi}{\varphi}\right)T_{20} - T_{21} + \frac{1}{2}\left(\frac{\pi}{\varphi^2}\right)\left(\frac{315}{64\varphi^2} + S_{47}\right) - \frac{55}{3\varphi^4} + S_{18}\right] \quad (8.11c)$$

9. DYNAMIC BUCKLING LOAD, λ_D

Having determined the maximum deflection, w_a as in (8.11a – c), we shall now determine the dynamic buckling load λ_D from the maximization (7.1). We shall first reverse the series (8.10a) which we now write as

$$w_a = \epsilon e_1 + \epsilon^3 e_4 + \cdots \quad (9.1a)$$

where,

$$e_1 = 4\left[1 - \frac{\delta\pi}{2\varphi} + \frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2\right] \quad (9.1b)$$

$$e_3 = \frac{30\alpha B^3}{\varphi^3}\left[\left(1 + \frac{\varphi^2 S_9}{15}\right) + A_{31}\delta + A_{32}\delta^2\right] \quad (9.1c)$$

Thus, for the reversal, we write

$$\epsilon = f_1 w_a + f_3 w_a^3 + \cdots \quad (9.2)$$

Upon substituting in (8.13) for w_a from (8.12a), and equating the coefficients of ϵ, we get

$$f_1 = \frac{1}{e_1}, \quad f_3 = -\left(\frac{e_3}{e_1^4}\right) \quad (9.3a)$$

The maximization (7.1) is now easily executed from (9.2) to yield

$$f_1 + 3f_3 w_{aD}^3 = 0, \quad (9.3b)$$

where w_{aD} is the maximum value of the deflection at dynamic buckling. Thus, we have

$$w_{aD} = \sqrt{\frac{-f_1}{3f_3}} \quad (9.4)$$

On substituting from (9.3a) in (9.4) we get

$$w_{aD} = \frac{1}{\sqrt{3}}\left(\frac{e_1^3}{e_3}\right) \quad (9.5)$$

If we next evaluate (9.2) at dynamic buckling, we get

$$\epsilon = f_1 w_{aD} + f_3 w_{aD}^3 + \cdots = w_{aD}(f_1 + 3f_3 w_{aD}^2) \quad (9.6)$$

On substituting in (9.6) for f_1, f_3 *and* w_{aD}, we get

$$\epsilon = \frac{2}{3\sqrt{3}}\left(\frac{e_1}{e_3}\right)^{\frac{1}{2}} \quad (9.7)$$

On substituting in (9.7), we get the equation for determining the dynamic buckling load λ_D as

$$(16m^4 - 8m^2\lambda_D + 1)^{\frac{3}{2}} = 18\sqrt{10}(\epsilon\bar{a}_m)m^2\lambda_D\alpha^{\frac{1}{2}}\left[\frac{\left[\left(1+\frac{\varphi^2 S_9}{15}\right)+A_{31}\delta+A_{32}\delta^2\right]}{1-\frac{\delta\pi}{2\varphi}+\frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2}\right]^{\frac{1}{2}} \quad (9.9)$$

Equation (9.9) gives an implicit formula for determining the dynamic buckling load λ_D. By using equation (5.25), we can eliminate the imperfection parameter ϵ and hence relate λ_S to λ_D. This gives

$$\left(\frac{16m^4-8m^2\lambda_D+1}{16m^4-8m^2\lambda_S+1}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[\frac{\frac{\left[\left(1+\frac{\varphi^2 S_9}{15}\right)+A_{31}\delta+A_{32}\delta^2\right]}{1-\frac{\delta\pi}{2\varphi}+\frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2}}{1+\frac{4}{15}\left(\frac{16m^4-8m^2\lambda_S+1}{1296m^4-72m^2\lambda_S+1}\right)}\right]^{\frac{1}{2}} \quad (9.10)$$

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421) I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

The least value of the dynamic buckling load λ_D is obtained when $m = 1$ and for this value, equations (9.9) and (9.10) respectively become

$$(17 - 8\lambda_D)^{\frac{3}{2}} = 18\sqrt{10}(\epsilon\bar{a}_1)\lambda_D\alpha^{\frac{1}{2}}\left[\frac{\left[\left(1+\frac{\varphi^2 s_9}{15}\right)+A_{31}\delta+A_{32}\delta^2\right]}{1-\frac{\delta\pi}{2\varphi}+\frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2}\right]^{\frac{1}{2}} \quad (9.11)$$

and

$$\left(\frac{17-8\lambda_D}{17-8\lambda_S}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[\frac{\frac{\left[\left(1+\frac{\varphi^2 s_9}{15}\right)+A_{31}\delta+A_{32}\delta^2\right]}{1-\frac{\delta\pi}{2\varphi}+\frac{1}{4}\left(\frac{\delta\pi}{\varphi}\right)^2}}{1+\frac{4}{15}\left(\frac{17-8\lambda_S}{1297-72\lambda_S}\right)}\right]^{\frac{1}{2}} \quad (9.12)$$

where the right hand sides of (9.11) and (9.12) are to be evaluated at $m = 1$.

10. ANALYSIS OF RESULTS

The graphical plots of the results were done using Q-Basic codes and the results are hereby presented in Fig. 1, Fig. 2 and Fig. 3.

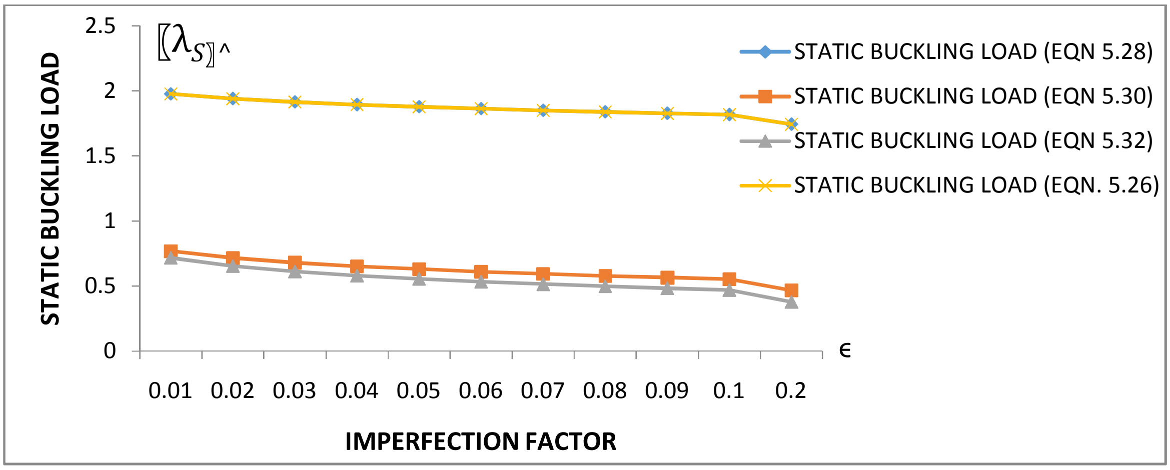

Fig. 1: Relationship between the Static buckling load, λ_S and Imperfection factor, ϵ using Eqn. (5.26), Eqn.(5.28), Eqn.(5.30) and Eqn.(5.32).

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
I.U. Udo-Akpan[1] *
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

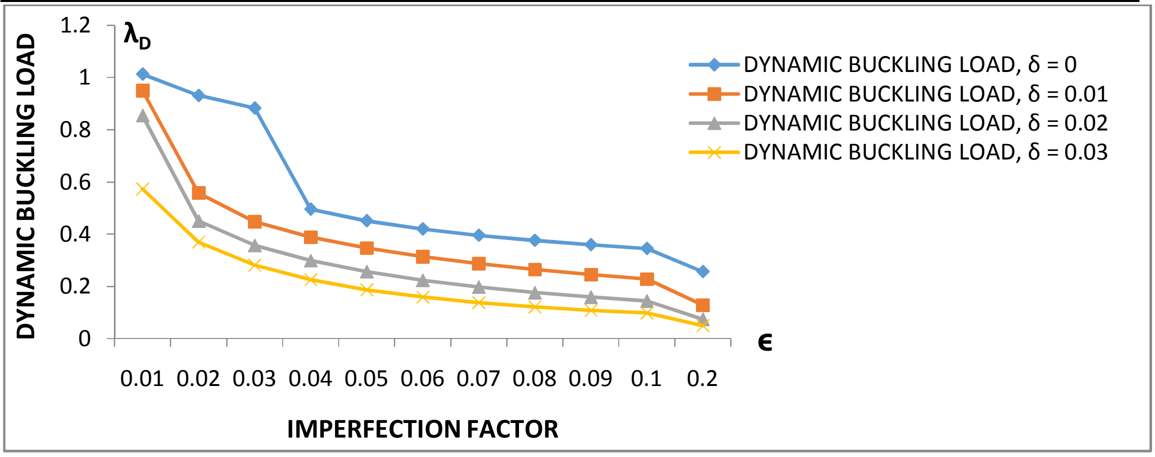

Fig. 2: Relationship between the Dynamic buckling loads and Imperfection parameters at some fixed values of the damping factor, δ, using Eqn. (9.11).

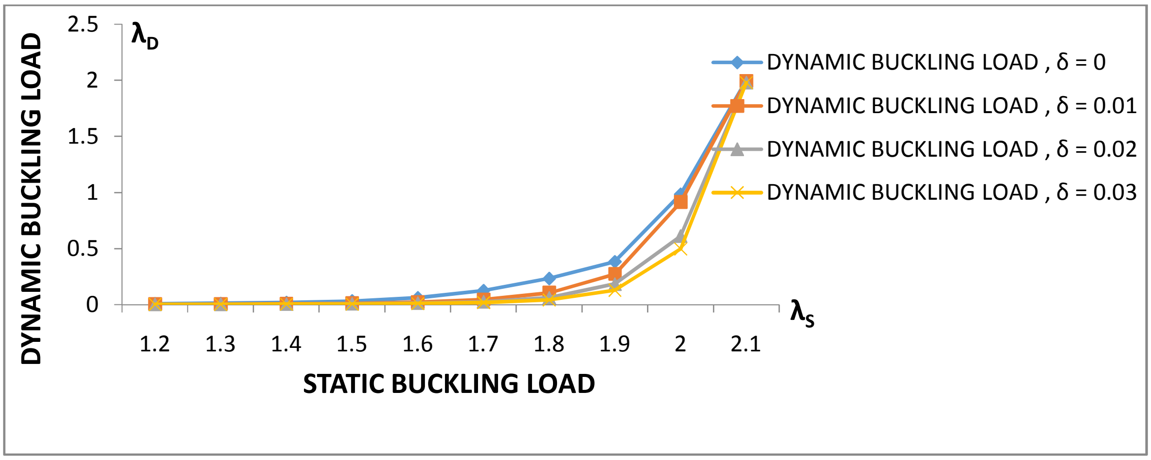

Fig. 3: Relationship between the Dynamic buckling loads and the Static buckling loads at some fixed values of the damping factor, δ, using Eqn. (9.12).

From Fig. 1, we observe that the static buckling load of a clamped column is always higher than that of the same column with simply–supported boundary conditions. In general, the static buckling load of a column with either clamped or simply-supported boundary conditions and whose deflections are strictly in the shape of imperfection always has the least static buckling load. However, while the static buckling load of a clamped column satisfies the inequality, $1 < \lambda_S < 2.125$, a similar column with simply–supported boundary supports satisfies the inequality $0 < \lambda_S < 1$.
Fig. 2 shows that the dynamic buckling load λ_D decreases with increased imperfection for any value of the damping parameter.
From Fig. 3, we observe that at low values of the static buckling load λ_S (precisely for $1 < \lambda_S < 1.45$), there is no significant difference in the value of the dynamic buckling load λ_D of the column, whether damped or undamped. However, at higher values of λ_S (i.e, $1.45 < \lambda_S < 2.125$), the dynamic buckling load λ_D rises with λ_S and the highest of

such rise is the undamped case. It is not clear whether such a result is specific to clamped boundary conditions or whether it is general. As in the static loading case, the inequality satisfied by the clamped column is $1 < \lambda_S < 2.125$. We thus observe that generally, whether in the static or dynamic loading cases, clamped columns buckle at much more higher loads than similar columns with simply–supported boundary supports. For reasons attributed to nonlinearity and imperfection, clamped columns on nonlinear elastic foundations buckle at lower values of buckling loads than the corresponding classical buckling load of the column.

We observe that while the buckling modes split into three distinct modes proportional to $(1 - \cos 2mx)$, $(1 - \cos 4mx)$ and $(1 - \cos 6mx)$, it is only the buckling modes in the shapes of $(1 - \cos 2mx)$ and $(1 - \cos 6mx)$ that eventually contribute to dynamic buckling. The mode in the shape of $(1 - \cos 4mx)$ does not contribute.

Lastly, as seen in equations (8.20) and (8.22), we are able to directly relate λ_D to λ_S even without the knowledge of the size of the associated imperfection. In this way, we have circumvented the process of repeating the arduous manipulations for different imperfection parameters.

11. CONCLUSION

We have carried out an analytical investigation of the buckling of an imperfect clamped column lying on a nonlinear elastic foundation but struck axially by a step load. It is our contention that similar enquiries can be extended to other loading conditions apart from step load and to other structures apart from columns.

ACKNOWLEDGEMENTS:

Authors acknowledge the valuable contribution their mentor, Prof. A.M. Ette (FNMS), for his guardian and support on this work.

REFERENCES

[1] Timoshenko, S.P. and Gere, J.M., *Theory of elastic stability*, Dover Publications, London. 2012.

[2] Thompson, J. M. T. and Hunt, G. W.,*A general theory of elastic stability*, John Wiley and Sons Ltd., London. 1973.

[3] Hu, N. and Burgueño, R.,*Elastic postbuckling response of axially – loaded cylindrical shells with seeded geometric imperfection design*; Thin – walled structures, 96, 256 – 268, 2015.

[4] Amazigo, J. C., Budiansky, B. and Carrier, G. F.,*Asymptotic Analyses of the buckling of imperfect columns on nonlinear elastic foundations*, Int. J. Solids Structures, 10, 1342 – 1356, 1970.

[5] Qiang, H., Zhang, S. and Yang, G.,*The Asymptotic solution of a dynamic buckling problem in elastic column*, Appl. Maths. and Mech. (English edition)20(8), 867 – 872, 1999.

[6] Amazigo, J. C. and Frank, D.,*Dynamic buckling of an imperfect column on a nonlinear foundation*, Quart. Appl. Math., 31(6), 1 – 9, 1973.

[7] Artem, H. S. and Aydin, L.,*Exact solution and dynamic buckling analysis of a beam – column loading*. Appl. Math. and Mech. 31(10), 1317 – 1324, 2010.

[8] Ette, A. M., Chukwuchekwa, J. U. and Udo – Akpan, I. U.,*On the buckling of a clamped viscously damped column trapped by a step load*. Int. J. of Appl. Sciences and Mathematics, 3 (2), 117 – 123, 2016.

[9] Udo-Akpan, I.U. and Ette, A.M.,*On the dynamic buckling of a model structure with quadratic nonlinearity struck by a step load superposed on a quasi – static load*, J. of the Nigerian Assoc. of Math. Physics, 35, 461-472, 2016.

[10] Ette, A.M. and Udo-Akpan, I.U.,*On the buckling of a pre-statically loaded but viscously damped quadratic model structure trapped by a step load*, Global J. of Mathematics, 7 (2), 761 – 770, 2016.

[11] Chukwuchekwa, J. U. and Ette, A. M.,*Asymptotic analysis of an improved quadratic model structures subjected to static loading*. J. of Nigerian Assoc. of Math. Physics, 32, 237-244, 2015.

[12] Chukwuchekwa, J. U. and Ette, A. M.,*Perturbation technique in the buckling of a viscously damped elastic model structure pre – statically loaded but struck by a step load*, J. of Nigerian Assoc. of Math. Physics, 37 – 70, 2016.

[13] Kolakowski, Z.,*Static and dynamic interactive buckling regarding axial extension mode of thin – walled channels*, J. of Theoritical and Appl. Mech. 48 (3), 703 – 714, 2010.

[14] Kowal – Michalska, K.,*About some important parameters in dynamic buckling analysis of plate structures subjected to pulse loading*, J. of Mech. And Mechanical Engng., 14 (2), 259 – 279, 2010.

[15] Mcshane, G.J., Pingle, S. M., Deshpande, V.S. and Flock, N.A.,*Dynamic buckling of inclined structures, Int. J. of Solids and Struct.,* 49, 2830 – 2838, 2012.

[16] Belyaev, A. K. and gLim, D. N.,*Stability of transverse vibration of rod under longitudinal step – wise loading*, J. of Physics, Conference Series, 451, 1 – 6, 2012.

[17] Priyadarsini, R.S., Kalyanaraman, V. and Srinvasan, S.M.,*Numerical and experimental study of buckling of advanced fibre composite cylinder under axial compression*, Int. J. of Structural stability and Dynamics, 12 (4), 1 – 24, 2012.

[18] Budiansky, B. and Hutchinson, J.W.,*Dynamic buckling of imperfection – sensitive structures,*Proceedings of XI International Congr. Applied Mechanics. Springer – Verlag, Berlin. 1966.

IJO - INTERNATIONAL JOURNAL OF MATHEMATICS
(ISSN: 2992-4421)
https://ijojournals.com/ Volume 07 Issue 02 || February, 2024 ||

I.U. Udo-Akpan[1] *

[19] Ette, A. M. and Chukwuchekwa, J. U., *On the static buckling of an externally pressurized finite circular cylindrical shell*, J. of Nigerian Assoc. of Math. Physics, 11, 323 – 332, 2007.

[20] A. M. Ette, J. U. Chukwuchekwa, **I. U. Udo-Akpan** and Onuoha, N.O. (2020); A Multi-Timing Perturbation Analysis of the deformation and Dynamic Buckling of a Viscously Damped Toroidal Shell Segment Stressed by a Step Load, *IOSR Journal of Mathematics (IOSR-JM)*, 16(4), 40-59.

[21] A. M. Ette, J. U. Chukwuchekwa, N.O. Onuoha, and **I. U. Udo-Akpan** (2020); On the Deformation and Static Buckling of a Toroidal Shell Segment using a two-term Fourier series imperfections, *International Journal of Mathematics Trends and Technology (IJMTT),* 66(8), 100-114

[22] G. Ozoigbo, A.M. Ette, J. Chukwuchekwa, W. Osuji, **I.U. Udo-Akpan** (2022); On the Analysis of a Pre-statically Loaded Nonlinear Cubic structure Pressurized by an Explicitly Time Dependent SlowlyVarying Load, American Journal of Mechanics and applications, 10(1), 1-15

[23] G.E. Ozoigbo, A.M. Ette, J.U. Chukwuchekwa, W.I. Osuji, **I.U. Udo-Akpan** (2023); Nonlinear Analytical Investigation of Dynamic Buckling of Spherical Shell Trapped Under a Periodic load, Mechanics of Solids, Springer Link, 58, 202-215.

[24] Obong, H.P. and UdoAkpan, I.U. (2023); A Review of Tensor Interaction in the Theory of Potential Models, Scientia Africana, 22(3), 11-22.

[25] O.G. Udoaka, U.J. Etim and I.U. UdoAkpan, (2024) Efficient Solution of Nonhomogeneous Linear Differential Equations with Constant coefficients: the method of undetermined coefficient approach, IEEE-SEM, 12(1), 21-36