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Techniques to solve Diophantine Equation of Degree Ten with Six Unknowns

$$x^6 - y^6 - 3456z^3 = 800(p^2 - q^2)R^8$$

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Abstract:

This paper focuses on finding varieties of distinct integer solutions to the Diophantine equation of degree ten with six unknowns given by $x^6 - y^6 - 3456z^3 = 800(p^2 - q^2)R^8$ *through employing the substitution strategy and method of factorization. A new representation for the factorization of integer 40 involving sides of Pythagorean triangle has been introduced.*

Key words: Higher degree Diophantine equation, Integer solution.

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Introduction:

It is well-known that the subject of Diophantine equations occupies a pivotal role in the Number theory. There is a vast general theory for higher degree Diophantine equations in many variables and it is a topic for research even today. While collecting problems on Diophantine equations of degree ten with six unknowns, the following paper [1] has been noticed and the authors have presented only few sets of integer solutions. It is worth to mention that the equation presented in [1] has many more fascinating patterns of solutions in integers.

In this paper, the process of obtaining some more choices of solutions in integers to the Diophantine equation in title is illustrated. Substitution strategy and factorization method are applied successfully to obtain different choices of integer solutions to the considered equation. It is to be noted that a new representation for the factorization of integer 40 involving sides of Pythagorean triangle has been introduced.

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Method of Analysis:

The Diophantine Equation is

$$x^6 - y^6 - 3456z^3 = 800(p^2 - q^2)R^8 \tag{1}$$

To start with, it is observed by inspection that (1) is satisfied by the following sextuples {x, y, z, p, q, R} = $\{8u^{4v}, 4u^{4v}, 2u^{8v}, 18u^{4v}, -6u^{4v}, u^{2v}\}$, $\{8*40^2*(9a^2+4b^2)^2, 4*40^2*(9a^2+4b^2)^2, 2*40^4*(9a^2+4b^2)^4, 18*40^2*(9a^2+4b^2)^2, -6*40^2*(9a^2+4b^2)^2, 40*(9a^2+4b^2)\}$, $\{32w^2, 16w^2, 32w^2, 72w^2, 24w^2, 2w\}$. However there are often fascinating solution patterns to (1) that are illustrated as follows.

Introduction of the transformation

$$x = 3s + 2t, y = 3s - 2t, z = st, p = 6t + 6s, q = 6t - 6s \tag{2}$$

in (1) leads to

$$9s^2 + 4t^2 = 40R^4 \tag{3}$$

Solving (3) through different ways and using (2), one obtains many non-zero distinct integer solutions to (1).

Pattern 1:

The choice

$$t = R \tag{4}$$

in leads to

$$s^2 = 4R^2\left(\frac{10R^2 - 1}{9}\right) \tag{5}$$

Let

$$10\,R^2 = 9\alpha^2 + 1 \tag{6}$$

The smallest positive integers to (6) is

$$\alpha_0 = 1,\ R_0 = 1$$

Let $R_1 = h - R_0$, $\alpha_1 = h + \alpha_0$ (7)

be the second solution to (6). Substituting (7) and (6) and performing some algebra, we get

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$$h = 20R_0 + 18\alpha_0$$

In view of (7), we get

$$R_1 = 19R_0 + 18\alpha_0 \ , \ \alpha_1 = 20R_0 + 19\alpha_0$$

which is written in matrix form as

$$(R_1, \alpha_1)^t = M(R_0 + \alpha_0)^t$$

where $M = \begin{pmatrix} 19 & 18 \\ 20 & 19 \end{pmatrix}$ and t is the transpose.

The repetition of the above process leads to the general solution (R_n, α_n) to (6) as

$$(R_n, \alpha_n)^t = M^n (R_0, \alpha_0)^t$$

Let $\tilde{\alpha}$, $\tilde{\beta}$ be the eigen values of the matrix M. Then, it is found that

$$\tilde{\alpha} = 19 + 6\sqrt{10}, \ \tilde{\beta} = 19 - 6\sqrt{10}$$

It is well known that

$$M^n = \frac{\tilde{\alpha}^n}{\tilde{\alpha} - \tilde{\beta}}(M - \tilde{\beta} I) + \frac{\tilde{\beta}^n}{\tilde{\beta} - \tilde{\alpha}}(M - \tilde{\alpha} I)$$

where I is the unit matrix of order 2.

Thus, $(R_n, \alpha_n)^t = \begin{pmatrix} \dfrac{\tilde{\alpha}^n + \tilde{\beta}^n}{2} & \dfrac{3(\tilde{\alpha}^n - \tilde{\beta}^n)}{2\sqrt{10}} \\ \dfrac{5(\tilde{\alpha}^n - \tilde{\beta}^n)}{3\sqrt{10}} & \dfrac{\tilde{\alpha}^n + \tilde{\beta}^n}{2} \end{pmatrix} (R_0, \alpha_0)^t$

From (5), we have

$$(R_n, \alpha_n)^t = M^n (R_0, \alpha_0)^t$$

From (5), we have

$$s_n = 2R_n \alpha_n$$

In view of (2), the corresponding integer solutions to (1) are given by

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$$x_n = 2R_n(3\alpha_n + 1)$$
$$y_n = 2R_n(3\alpha_n - 1)$$
$$z_n = 2R^2{}_n\alpha_n$$
$$p_n = 6R_n(1+2\alpha_n)$$
$$q_n = 6R_n(1-2\alpha_n)$$
$$t_n = R_n$$

where

$$R_n = \frac{\tilde{\alpha}^n + \tilde{\beta}^n}{2} + \frac{3(\tilde{\alpha}^n - \tilde{\beta}^n)}{2\sqrt{10}},$$
$$\alpha_n = \frac{5(\tilde{\alpha}^n - \tilde{\beta}^n)}{3\sqrt{10}} + \frac{\tilde{\alpha}^n + \tilde{\beta}^n}{2}.$$

Pattern 2:

Taking

$$s = 2kR \tag{8}$$

in (3), it is written as

$$t^2 = R^2(10R^2 - 9k^2) \tag{9}$$

Let

$$\alpha^2 = (10R^2 - 9k^2) \tag{10}$$

which is satisfied by

$$R_0 = k,\ \alpha_0 = k$$

To obtain the other solutions to (10), consider the corresponding pell equation

$$\alpha^2 = 10\,R^2 + 1$$

whose general solution $(\tilde{R}_n, \tilde{\alpha}_n)$ is given by

$$\tilde{R}_n = \frac{1}{2\sqrt{10}} g_n, \quad \tilde{\alpha}_n = \frac{1}{2} f_n \tag{11}$$

Applying the lemma of Brahmagupta between (R_0, α_0) and $(\tilde{R}_n, \tilde{\alpha}_n)$, the general solution (R_{n+1}, α_{n+1}) to (10) is given by

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$$R_{n+1}=\frac{k}{2}f_n+\frac{k}{2\sqrt{10}}g_n, \tag{12}$$

$$\alpha_{n+1}=\frac{k}{2}f_n+\frac{5k}{\sqrt{10}}g_n.$$

From (8) and (9), we get

$$s_{n+1}=2kR_{n+1}$$
$$t_{n+1}=R_{n+1}+\alpha_{n+1}$$

In view of (2), the corresponding integer solutions to (1) are given by,

$$x_{n+1}=\frac{k}{2\sqrt{10}}(\sqrt{10}f_n+g_n)(6k+2),$$
$$y_{n+1}=\frac{k}{2\sqrt{10}}(\sqrt{10}f_n+g_n)(6k-2),$$
$$z_{n+1}=\frac{k^4}{40\sqrt{10}}(\sqrt{10}f_n+g_n)^2(\sqrt{10}f_n+10g_n),$$
$$p_{n+1}=3(\frac{k}{2}f_n+\frac{k}{2\sqrt{10}}g_n)(\alpha_{n+1}+2k),$$
$$q_{n+1}=3(\frac{k}{2}f_n+\frac{k}{2\sqrt{10}}g_n)(\alpha_{n+1}-2k)$$

Jointly with (12), where

$$f_n=(19+6\sqrt{10})^{n+1}+(19-6\sqrt{10})^{n+1},$$
$$g_n=(19+6\sqrt{10})^{n+1}-(19-6\sqrt{10})^{n+1}.$$

Pattern 3:

Assume

$$R=9u^2+9v^2 \tag{13}$$

Consider

$$40=(2+6i)(2-6i) \tag{14}$$

Substituting (13), (14) in (3) and using the method of factorizations, we have

$$(3s+i2t)=(2+6i)(3u+i3v)^4$$

Equating real and imaginary parts, we have

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$$s = 3^3[2u^4 - 12u^2v^2 + 2v^4 - 24uv(u^4 - v^4)]$$
$$t = 3^4[3u^4 - 18u^2v^2 + 3v^4 + 4uv(u^4 - v^4)]$$

In view of (2), we get

$$x = 3^3[8u^4 - 48u^2v^2 + 8v^4 - 16uv(u^2 - v^2)]$$
$$y = 3^4[-4u^4 + 24u^2v^2 - 4v^4 - 32uv(u^2 - v^2)]$$
$$z = 3^7[3u^4 - 18u^2v^2 + 3v^4 + 4uv(u^2 - v^2)]$$
$$p = 6*3^7[6u^4 - 30u^2v^2 + 5v^4 - 20uv(u^2 - v^2)]$$
$$q = 6*3^7[u^4 - 6u^2v^2 + v^4 + 28uv(u^2 - v^2)]$$

Pattern 4:

It is worth to mention that the integer 40 may also be written as

$$40 = \frac{[2f(\alpha,\beta) + i2g(\alpha,\beta)][2(\alpha,\beta) - i2g(\alpha,\beta)]}{(\alpha^2 + \beta^2)^2} \tag{15}$$

where $f(\alpha,\beta) = 3(\alpha^2 - \beta^2) + 2\alpha\beta$; $g(\alpha,\beta) = 3(2\alpha\beta) + (\alpha^2 - \beta^2)$

Substituting (13) & (15) in (3) and employing the method of factorization, we get

$$3s + i2t = \frac{3^4}{(\alpha^2 + \beta^2)}[2f(\alpha,\beta) + i2g(\alpha,\beta)][F(u,v) + iG(u,v)] \tag{16}$$

where

$$F(u,v) = u^4 - 6u^2v^2 + v^4$$
$$G(u,v) = 4uv(u^2 - v^2)$$

Equating the real and imaginary part of (16), one obtains

$$\left.\begin{aligned} s &= \frac{3^3}{(\alpha^2 + \beta^2)}[2f(\alpha,\beta)F(u,v) - 2g(\alpha,\beta)G(u,v)] \\ t &= \frac{3^4}{(\alpha^2 + \beta^2)}[f(\alpha,\beta)G(u,v) + g(\alpha,\beta)F(u,v)] \end{aligned}\right. \tag{17}$$

Replacing u by $(\alpha^2 + \beta^2)u$, v by $(\alpha^2 + \beta^2)v$ in (16), we have

$$\left.\begin{aligned} s &= 3^3(\alpha^2 + \beta^2)^3[2f(\alpha,\beta)F(u,v) - 2g(\alpha,\beta)G(u,v)] \\ t &= 3^4(\alpha^2 + \beta^2)^3[f(\alpha,\beta)G(u,v) + g(\alpha,\beta)F(u,v)] \end{aligned}\right. \tag{18}$$

Also, from (13), $R = 9(\alpha^2 + \beta^2)^2(u^2 + v^2)$ (19)

From (2) and (18), it is seen that

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$$x = 3^4(\alpha^2+\beta^2)^3[2f(\alpha,\beta)\{F(u,v)+G(u,v)\}+2g(\alpha,\beta)\{F(u,v)-G(u,v)\}]$$
$$y = 3^4(\alpha^2+\beta^2)^3[2f(\alpha,\beta)\{F(u,v)-G(u,v)\}+2g(\alpha,\beta)\{F(u,v)+G(u,v)\}]$$
$$z = 3^7(\alpha^2+\beta^2)^6[2f(\alpha,\beta)F(u,v)-2g(\alpha,\beta)G(u,v)][f(\alpha,\beta)G(u,v)+g(\alpha,\beta)F(u,v)]$$
$$p = 6*3^3(\alpha^2+\beta^2)^3[3f(\alpha,\beta)G(u,v)+3g(\alpha,\beta)F(u,v)+2f(\alpha,\beta)F(u,v)-2g(\alpha,\beta)G(u,v)]$$
$$q = 6*3^3(\alpha^2+\beta^2)^3[3f(\alpha,\beta)G(u,v)+3g(\alpha,\beta)F(u,v)-2f(\alpha,\beta)F(u,v)-2g(\alpha,\beta)G(u,v)]$$

which satisfy (1) along with (19).

Pattern 5:

Rewrite (3) as

$$9s^2+4t^2 = 40R^4*1 \tag{20}$$

Assume the integer 1 on the RHS of (1) as

$$1 = \frac{(p^2-q^2+i2pq)(p^2-q^2-i2pq)}{(p^2+q^2)} \tag{21}$$

Substituting (14), (21) in (20), we have

$$(3s+i2t) = (2+6i)\frac{(p^2-q^2+i2pq)}{(p^2+q^2)}(3s+i3v)^4$$
$$\Rightarrow (3s+i2t) = \frac{3^4(2+6i)}{(p^2+q^2)}[F(u,v)+iG(u,v)][J(p,q)+iK(p,q)]$$
$$\Rightarrow (3s+i2t) = \frac{3^4(2+6i)}{(p^2+q^2)}[P(u,v,p,q)+iQ(u,v,p,q)]$$

where

$$J(p,q) = p^2-q^2;\ K(p,q) = 2pq$$
$$F(u,v) = u^4-6u^2v^2+v^4;\ G(u,v) = 4uv(u^2-v^2)$$
$$P(u,v,p,q) = F(u,v)J(p,q)-G(u,v)K(p,q)$$
$$Q(u,v,p,q) = F(u,v)K(p,q)+G(u,v)J(p,q)$$

Equating real and imaginary part of the above equation, we have

$$s = \frac{3^3}{(p^2+q^2)}[2P(u,v,p,q)-6Q(u,v,p,q)]$$
$$t = \frac{3^4}{(p^2+q^2)}[3P(u,v,p,q)+Q(u,v,p,q)]$$

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Replacing u by (p^2+q^2) A, v by (p^2+q^2) B in the values of s, t and R, we have

$$s = 27(p^2+q^2)^3[2P(A,B,p,q) - 6Q(A,B,p,q)]$$
$$t = 81(p^2+q^2)^3[3P(A,B,p,q) + Q(A,B,p,q)]$$
$$R = 9(p^2+q^2)^2(A^2+B^2)$$

Conclusion:

It has been shown that higher degree Diophantine equation with multiple variables may be solved through strategies like substitution and factorization by reducing it to a lesser degree solvable Diophantine equation. One may search for other choices of higher degree Diophantine equations with multiple variables for obtaining their respective integer solutions.

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