Algebraic Solution to Simultaneous Linear First-Order Non-Homogeneous Differential Equations with Constant Coefficients

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ABSTRACT

Ordinary differential equations, both first and second order, are essential in the modeling of many physical systems. A system of simultaneous differential equations results from more complicated modeling involving more than one dependent variables with respect to a single independent variable. There are several methods in solving a system of simultaneous linear differential equations including variable substitution, Laplace transform and using the D-operator.

Proposed in this paper is a simplified method of solving a set of two non-homogeneous linear first-order simultaneous ordinary differential equations with constant coefficients that falls into a certain form. An algebraic formula is developed to compute the solution to the said differential equations provided a certain necessary condition is satisfied.

Four different forms of the functions of the independent variable on the right side of the equations, namely constants, linear functions, natural exponential functions and sinusoidal functions, are considered. For each case, an algebraic formula to calculate the dependent variable as well as its derivative are developed. Moreover, these simple algebraic formulae can be easily programmed into spreadsheet where one just has to enter the values of the constants and coefficients from the original equations and instantly obtain the correct answers.

Key words: Simultaneous differential equations; Non-homogeneous differential equations; Cramer's rule.

1. Introduction

Ordinary Differential Equations (ODE) play a crucial role in understanding and modeling various physical phenomena across many fields such as engineering, sciences and economics. Among the numerous applications of first-order differential equations are solutions mixing, population change, heating or cooling, free-falling body, fluid dynamics, resistive-capacitive (RC) and resistive-inductive (RL) circuits. Second-order differential equations find its applications in spring-mass vibration, RLC circuits, wave propagations, etc. By solving a differential equation using its known initial conditions, which is known as Initial Value Problem (IVP), we can anticipate the behavior of the physical system over time. [7]

Simultaneous differential equations is a system of at least two differential equations with two or more dependent variables but share a single common independent variable. These equations govern the inter-relationship between the rate of change of the dependent variables with respect to the independent variable. [8]

The general form of an n^{th}-order linear Ordinary Differential Equation is:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \tag{1}$$

Since this paper only consider 1^{st}-order ODE, equation (1) reduces to

$$a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \tag{2}$$

where $a_1(x)$, $a_2(x)$ and $F(x)$ are functions of *x* only, and there is no restriction on the nature of these *x*-dependencies. [5, 6]

2 Example of a System of Simultaneous 1^{st}-order ODE

2.1 Double Mixing Problem

Figure 1 below shows a brine solution with concentration $B_1(kg/l)$ flows at a constant rate of $F_1(l/s)$ into Tank 1 that initially contains $V_1(l)$ of the same brine. The solution in the tank is kept well stirred and flows out of Tank 1 at a rate of F_2 with concentration B_2 directly into Tank 2 with volume V_2. The solution in Tank 2 is also well mixed and flows out at a rate of F_3 with concentration B_3. The mass of salt in Tank 1 and Tank 2 are $m_1(t)$ and $m_2(t)$ respectively.

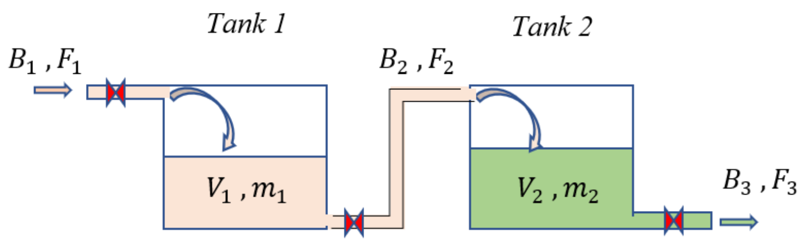

Figure 1: Double Mixing Problem

The IVP for this problem is given by:

$$\frac{dm_1}{dt} = B_1F_1 - \frac{m_1}{V_1+(F_1-F_2)\cdot t}F_2 \quad , \quad m_1(0) = a \tag{3a}$$

$$\frac{dm_2}{dt} = \frac{m_1}{V_1+(F_1-F_2)\cdot t}F_2 - \frac{m_2}{V_2+(F_2-F_3)\cdot t}F_3 \quad , \quad m_2(0) = b \tag{3b}$$

where *a* and *b* are the initial mass of salt in Tank 1 and Tank 2, respectively.

2.2 Simplifying IVP

In most applications, F_1, F_2 and F_3 are constant and equal, otherwise the volume of liquid in the two tanks would be changing with time. If the two volumes are assumed to be equal, that is $V_1 = V_2 = V$, then the IVP in equation (3) is simplified to

$$m_1' = A - Km_1 \tag{4a}$$

$$m_2' = Km_1 - Km_2 \tag{4b}$$

where

$$A = B_1 F, \quad K = \frac{F}{V} \quad \text{and} \quad F_1 = F_2 = F_3 = F. \tag{4c}$$

Putting equation (4) in matrix form, we have

$$\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = \begin{bmatrix} -K & 0 \\ K & -K \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + \begin{bmatrix} A \\ 0 \end{bmatrix} \tag{5}$$

If the input concentration can be continuously adjusted to be in direct proportional to the varying mass of salt in Tank 1, that is $B_1 \propto m_1$, then

$$A = p m_1 F = C \cdot m_1 \tag{6}$$

where *p* is the constant of proportionality between B_1 and m_1, and the constant *c* is the product of *p* and *F*. With this assumption, equation (5) can be simplified to

$$\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = \begin{bmatrix} -K + C & 0 \\ K & -K \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \tag{7}$$

2.3 De-coupling of the System

The eigenvalues of equation (7) can be found by solving the characteristic polynomial

$$\begin{vmatrix} \lambda + K - C & 0 \\ -K & \lambda + K \end{vmatrix} = 0 \tag{8}$$

which yields

$$\lambda_1 = -K \quad \text{and} \quad \lambda_2 = -(K - C) \tag{9}$$

with the corresponding eigenvectors

$$\alpha_1 = r \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \alpha_2 = s \begin{bmatrix} \frac{C}{K} \\ 1 \end{bmatrix}, \quad r, s \in R. \tag{10}$$

The transition matrix *P* and its inverse is given by

$$P = \begin{bmatrix} 0 & \frac{C}{K} \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{-\frac{C}{K}} \begin{bmatrix} 1 & -\frac{C}{K} \\ -1 & 0 \end{bmatrix} \tag{11}$$

As expected,

$$\frac{1}{-\frac{C}{K}}\begin{bmatrix}1 & -\frac{C}{K}\\ -1 & 0\end{bmatrix}\begin{bmatrix}-K+C & 0\\ K & -K\end{bmatrix}\begin{bmatrix}0 & \frac{C}{K}\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \tag{12}$$

Let's define

$$m = Px \quad \text{and } m' = Px' \tag{13}$$

where $x = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ is a dummy function.

Substitute (13) into (7) yields

$$Px' = \begin{bmatrix}-K+C & 0\\ K & -K\end{bmatrix}Px \tag{14}$$

Multiply (14) by P^{-1} gives

$$P^{-1}Px' = P^{-1}\begin{bmatrix}-K+C & 0\\ K & -K\end{bmatrix}Px \tag{15}$$

Incorporate equations (9), (11) and (12) into (15), we have

$$\begin{bmatrix}x_1'(t)\\ x_2'(t)\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{16a}$$

which gives

$$x_1' = \lambda_1 x_1 \tag{16b}$$

$$x_2' = \lambda_2 x_2 \tag{16c}$$

The system is now decoupled [9, 10].

2.4 Solution to example

The solutions to equation (16) are easily obtained using separation of variables method as

$$x_1(t) = D_1 e^{-Kt} \tag{17a}$$

$$x_2(t) = D_2 e^{-(K-C)t} \tag{17b}$$

Since $m = Px$, we have

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{C}{K} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} D_1 e^{-Kt} \\ D_2 e^{-(K-C)t} \end{bmatrix}. \tag{18}$$

With the initial conditions $m_1(0) = a$ and $m_2(0) = b$, it is easy to verify that the solutions to the IVP in (3) are

$$m_1(t) = ae^{-(K-C)t} \tag{19a}$$

$$m_2(t) = \left(b - \frac{Ka}{C}\right) e^{-Kt} + \frac{Ka}{C} e^{-(K-C)t} \tag{19b}$$

3 Existing Solution Methods

There are several methods to solve simultaneous differential equations, each has it own restrictions on the form of the equations. [11, 12, 13, 14, 15, 16] These methods are:

(i) Substitution

(ii) Elimination

(iii) D-Operator

(iv) Laplace transform

(v) Numerical recursive calculations

(vi) Software package

4. Proposed Algebraic Solutions

A special case of simultaneous differential equations is focused here, where only one dependent variable *y*, and one independent variable *x*, are involved. However, the solution must satisfy two non-homogeneous differential equations simultaneously.

The main result of this paper is stated as a theorem below.

Theorem 1

The solutions to the set of non-homogeneous linear first-order simultaneous ordinary differential equations with constant coefficients in the form of

$$a_1 y' + a_0 y = f(x) \tag{20a}$$

$$b_1 y' + b_0 y = g(x) \tag{20b}$$

are given by

$$y(x) = \frac{g(x)a_1 - f(x)b_1}{a_1 b_0 - a_0 b_1} \tag{21a}$$

$$y'(x) = \frac{f(x)b_0 - g(x)a_0}{a_1 b_0 - a_0 b_1} \tag{21b}$$

with the necessary condition

$$a_1 g'(x) - b_1 f'(x) = b_0 f(x) - a_0 g(x). \tag{21c}$$

Proof

Write the given equations in matric form as

$$\begin{bmatrix} a_1 & a_0 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \tag{22}$$

Define the following three determinants

$$|A| = \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix} = a_1 b_0 - a_0 b_1 \tag{23a}$$

$$|A_1| = \begin{vmatrix} f(x) & a_0 \\ g(x) & b_0 \end{vmatrix} = b_0 f(x) - a_0 g(x) \tag{23b}$$

$$|A_2| = \begin{vmatrix} a_1 & f(x) \\ b_1 & g(x) \end{vmatrix} = a_1 g(x) - b_1 f(x) \tag{23c}$$

Using Cramer's Rule, the solutions are

$$y'(x) = \frac{b_0 f(x) - a_0 g(x)}{a_1 b_0 - a_0 b_1} \quad \text{and} \quad y(x) = \frac{a_1 g(x) - b_1 f(x)}{a_1 b_0 - a_0 b_1} \tag{24}$$

However, since $y(x)$ and $y'(x)$ are not constants and $y'(x)$ is obtained by differentiating $y(x)$ with respect to x, the following necessary condition must be satisfied:

$$a_1 g'(x) - b_1 f'(x) = b_0 f(x) - a_0 g(x) \quad \blacksquare \tag{25}$$

Note that the solution $y(x)$ can be obtained using simple algebraic computation.

Now let's look at the four common functions for $f(x)$ and $g(x)$.

4.1 Case 1: $f(x)$ and $g(x)$ are constants.

$f(x) = f$ and $g(x) = g$, where f and g are scalar constants.

The necessary condition becomes

$$0 = b_0 f - a_0 g \tag{26}$$

and the solutions are

$$y'(x) = \frac{b_0 f - a_0 g}{\alpha} = 0 \tag{27a}$$

$$y(x) = \frac{a_1 g - b_1 f}{\alpha} \tag{27b}$$

where the constant α is defined as

$$\alpha = a_1 b_0 - a_0 b_1 \tag{28}$$

Numerical Example 1

$$2y' + 3y = 6$$

$$5y' - y = -2$$

$a_1 = 2, a_0 = 3, b_1 = 5, b_0 = -1, f = 6$ and $g = -2$.

$b_0 f - a_0 g = (-1)(6) - (3)(-2) = -6 + 6 = 0$

$\alpha = (2)(-1) - (3)(5) = -17$

So,

$$y'(x) = \frac{(-1)(6)-(3)(-2)}{-17} = 0$$

$$y(x) = \frac{(2)(-2)-(5)(6)}{-17} = \frac{-34}{-17} = 2$$

Check:

Eq 1: $2y' + 3y = 2(0) + 3(2) = 6$ √

Eq 2: $5y' - y = 5(0) - (2) = -2$ √

4.2 Case 2: $f(x)$ and $g(x)$ are linear functions of x.

$$f(x) = m_1 x + m_0 \text{ and } g(x) = n_1 x + n_0. \tag{29}$$

The necessary condition becomes

$$a_1 n_1 - b_1 m_1 = b_0(m_1 x + m_0) - a_0(n_1 x + n_0)$$
$$= b_0 m_1 x + b_0 m_0 - a_0 n_1 x - a_0 n_0$$

$$= (b_0m_1 - a_0n_1)x + (b_0m_0 - a_0n_0) \quad (30)$$

Equating coefficients of like powers yield the following two necessary conditions.

$$b_0m_1 - a_0n_1 = 0 \quad (31a)$$

$$a_1n_1 - b_1m_1 = b_0m_0 - a_0n_0 \quad (31b)$$

The solutions are

$$y(x) = \frac{(n_1x+n_0)a_1-(m_1x+m_0)b_1}{\alpha}$$

$$= \frac{(a_1n_1-b_1m_1)x+(a_1n_0-b_1m_0)}{\alpha} \quad (32a)$$

$$y'(x) = \frac{(m_1x+m_0)b_0-(n_1x+n_0)a_0}{\alpha}$$

$$= \frac{(b_0m_1-a_0n_1)x+(b_0m_0-a_0n_0)}{\alpha} \quad (32b)$$

Numerical Example 2

$$y' + 3y = 6x + 2$$

$$-3y' + 4y = 8x - 6$$

$a_1 = 1, a_0 = 3, b_1 = -3, b_0 = 4, m_1 = 6, m_0 = 2, n_1 = 8\ , n_0 = -6$

Condition 1: $b_0m_1 - a_0n_1 = (4)(6) - (3)(8) = 0$ => Satisfied

Condition 2: $a_1n_1 - b_1m_1 = b_0m_0 - a_0n_0$

$$(1)(8) - (-3)(6) = (4)(2) - (3)(-6)$$

$$8 + 18 = 8 + 18$$ => Satisfied

$$\alpha = (1)(4) - (3)(-3) = 13$$

$$y(x) = \frac{[(1)(8)-(-3)(6)]x+[(1)(-6)-(-3)(2)]}{13}$$

$$= 2x$$

$$y'(x) = \frac{[(4)(6)-(3)(8)]x+[(4)(2)-(3)(-6)]}{13}$$

$$= 2$$

Check:

Eq 1: $y' + 3y = 2 + 3(2x) = 6x + 2$ √

Eq 2: $-3(2) + 4(2x) = 8x - 6 \quad \surd$

4.3 Case 3: $f(x)$ and $g(x)$ are exponential functions

$$f(x) = pe^{kx} \quad \text{and} \quad g(x) = qe^{kx} \quad , \quad p, q, k = \text{constant.} \tag{33}$$

The necessary condition is

$$a_1 qke^{kx} - b_1 pke^{kx} = b_0 pe^{kx} - a_0 qe^{kx} \tag{34}$$

which simplifies to

$$k(a_1 q - b_1 p) = (b_0 p - a_0 q). \tag{35}$$

Using the above theorem, the solution is given by

$$y(x) = \frac{a_1 qe^{kx} - b_1 pe^{k}}{a_1 b_0 - a_0 b_1} = \frac{a_1 q - b_1 p}{\alpha} \cdot e^{kx} \tag{36a}$$

$$y'(x) = \frac{b_0 pe^{kx} - a_0 qe^{kx}}{a_1 b_0 - a_0 b_1} = \frac{b_0 p - a_0 q}{\alpha} \cdot e^{kx} \tag{36b}$$

$$\alpha = a_1 b_0 - a_0 b_1 \tag{36c}$$

Numerical Example 3

$$2y' + 3y = 3e^{3x}$$

$$y' - 2y = \frac{1}{3} e^{3x}$$

$a_1 = 2, a_0 = 3, b_1 = 1, b_0 = -2, p = 3, q = \dfrac{1}{3}, k = 3$

$k(a_1 q - b_1 p) = 3\left(\frac{2}{3} - 3\right) = -7$; $b_0 p - a_0 q = -6 - 1 = -7$ => Condition satisfied

$\alpha = (2)(-2) - (3)(1) = -7$

$$y(x) = \frac{(2)\left(\frac{1}{3}\right) - (1)(3)}{-7} \cdot e^{3x} = \frac{1}{3} \cdot e^{3x}$$

$$y'(x) = \frac{(-2)(3) - (3)\left(\frac{1}{3}\right)}{-7} \cdot e^{3x} = e^{3x}$$

Check:

Eq 1: $2y' + 3y = 2(e^{3x}) + 3\left(\frac{1}{3} e^{3x}\right) = 3e^{3x} \ \surd$

Eq 2: $y' - 2y = (e^{3x}) - 2\left(\frac{1}{3} e^{3x}\right) = \frac{1}{3} e^{3x} \ \surd$

These calculations are programmed into an Excel spreadsheet as shown below *Figure 2*. One just have to enter the corresponding constants and coefficients to obtain the correct answer instantly. The necessary condition is also checked and the solution is verified automatically.

Numerical example:															
	2	y'	3	y	=	3	e^	3	x	a1 = 2	a0 = 3		p = 3		k = 3
	1	y'	-2	y	=	0.33	e^	3	x	b1 = 1	b0 = -2		q = 0.33		
										Necessary condition:		-7	=	-7	
$y(x)$ =		0.33	*e^(kx)								α = -7				
$y'(x)$ =		1	*e^(kx)												
Check:															
Eq 1:	3	=	3												
Eq 2:	0.33	=	0.33												

Figure 2: Spread sheet for Numerical Example 3

4.4 <u>Case4</u>: $f(x)$ and $g(x)$ are sine and cosine functions

$$f(x) = m\sin x + n\cos x \quad \text{and} \quad g(x) = p\sin x + q\cos x \tag{37}$$

The necessary condition is

$$a_1(p\cos x - q\sin x) - b_1(m\cos x - n\sin x) = b_0(m\sin x + n\cos x) - a_0(p\sin x + q\cos x)$$

$$\sin x\,(-a_1q + b_1n - b_0m + a_0p) + \cos x\,(a_1p - b_1m - b_0n + a_0q) = 0 \tag{38}$$

which gives

$$a_1q - a_0p = b_1n - b_0m \tag{39a}$$

$$a_1p + a_0q = b_1m + b_0n \tag{39b}$$

The solution is given by

$$y(x) = \frac{(a_1p - b_1m)\sin x + (a_1q - b_1n)\cos x}{\alpha} \tag{40a}$$

$$y'(x) = \frac{(b_0m - a_0p)\sin x + (b_0n - a_0q)\cos x}{\alpha} \tag{40b}$$

<u>Numerical Example 4</u>

$$116y' - 58y = -7\sin x + 55\cos x$$

$$35y' + 15y = 11\sin x + 12\cos x$$

$a_1 = 116, a_0 = -58, b_1 = 35, b_0 = 15, m = -7, n = 55, p = 11, q = 12$

$\alpha = a_1b_0 - a_0b_1 = 3770$

Necessary conditions:

(1): $a_1 q - a_0 p = b_1 n - b_0 m$

$(116)(12) - (-58)(11) = (35)(55) - (15)(-7)$

$2030 = 2030$ => Condition satisfied

(2): $a_1 p + a_0 q = b_1 m + b_0 n$

$(116)(11) + (-58)(12) = (35)(-7) + (15)(55)$

$580 = 580$ => Condition satisfied

$$y(x) = \frac{[(116)(11)-(35)(-7)]\sin x + [(116)(12)-(35)(55)]\cos x}{3770}$$

$$= 0.403 \sin x - 0.141 \cos x$$

$$y'(x) = \frac{[(15)(-7)-(-58)(11)]\sin x + [(15)(55)-(-58)(12)]\cos x}{3770}$$

$$= 0.141 \sin x + 0.403 \cos x$$

Check:

Eq 1: $116y' - 58y = (16.356 \sin x + 46.748 \cos x) - (23.374 \sin x - 8.178 \cos x)$

$= -7.018 \sin x + 54.926 \cos x \quad \surd$

Eq 2: $35y' + 15y = (4.935 \sin x + 14.105 \cos x) + (6.045 \sin x - 2.115 \cos x)$

$= 10.980 \sin x + 11.990 \cos x \quad \surd$

5. Conclusion

A set of two non-homogeneous linear first-order simultaneous ordinary differential equations with constant coefficients of a certain form are studied. Four different types of the forcing functions, namely constants, linear functions, natural exponential functions and sinusoidal functions, have been investigated. For each type, an algebraic equation to determine the solution and its derivative as well as the required necessary conditions are derived. Moreover, these simple algebraic formulae can be easily programmed into a spreadsheet which can automatically compute the solution, check the necessary condition and verify the answer.

6. Discussion

The theorem and formulae derived are applicable only to one special form of non-homogeneous linear first-order simultaneous ordinary differential equations with constant coefficients. This is indeed restrictive. Moreover, this method only works for the stated four cases of the forcing function and a stringent necessary condition must be satisfied. However, most of the existing methods have their restrictions too and cannot be used to solve any form of simultaneous differential equations with different combinations of the independent and dependent variables and their derivatives.

Despite its limitations, the proposed theorem and the resulting formulae have the advantage that they are algebraic and can be easily programmed into a spreadsheet that can perform repeated computations with different constants and coefficients effortlessly. The spreadsheet can also be programmed to check the necessary condition and verify the solution. The investigation employing this technique to different forms of simultaneous ODE and with different forcing functions would be an interesting endeavor.

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